MATH1010G University Mathematics Week 7: Second-Order Test and Taylor Series

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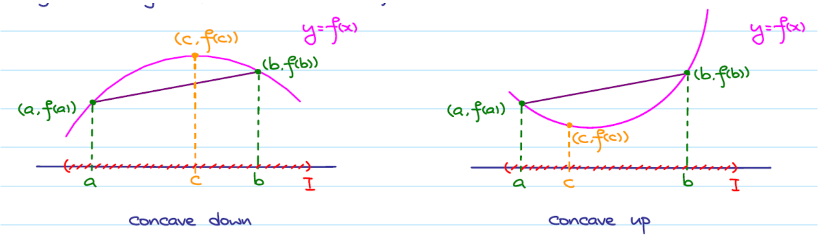
summary

- implicit differentiation
- linearization
- more on differentiability & continuity
- Rolle's theorem and mean value theorem
- monotonicity and first derivative check (relative max / min)
- Leibniz's rule

Convexity and second derivative check

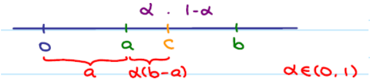
Definition

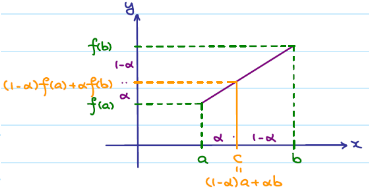
Let I be an interval and $f : I \to \mathbb{R}$. If for any $a, b, c \in I$ with $a < c < b$, $(c, f(c))$ is on or above (below) the line segment joining $(a, f(a))$ and $(b, f(b))$, f is **concave down (up)** on I.



Fact: If $a < c < b$, then for some $\alpha \in (0, 1)$,

$$c = a + \alpha(b - a) = (1 - \alpha)a + \alpha b$$



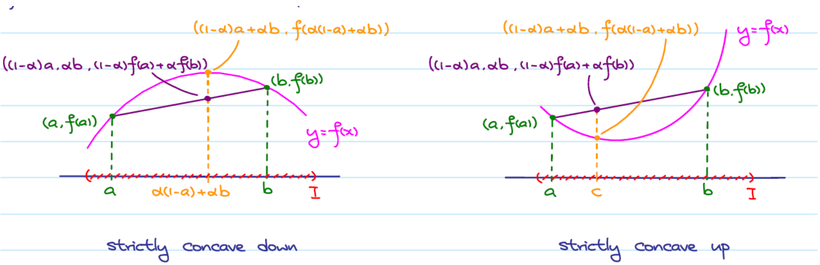


Definition (rephrase)

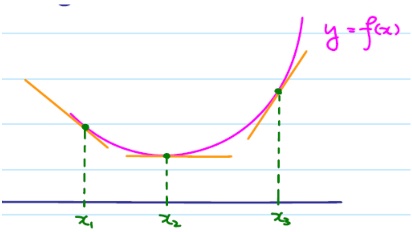
Let I be an interval and let $f : I \to \mathbb{R}$. If for any $a, b \in I$, $\alpha \in (0, 1)$,

$$f((1-\alpha)a + \alpha b) \geq (1-\alpha)f(a) + \alpha f(b)$$
$$(\text{or } f((1-\alpha)a + \alpha b) \leq (1-\alpha)f(a) + \alpha f(b)),$$

f is said to be concave down (up) on I.



$f''(x) > 0$ for $x \in I \Rightarrow f'(x)$ is strictly increasing.



Slope of the tangent line at $(x, f(x))$ increases as x increases!

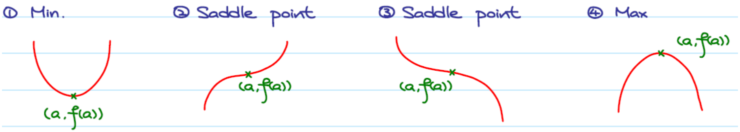
Theorem

Let I be an interval. If $f''(x) \geq 0$ ($f''(x) \leq 0$) for all $x \in I$, then $f(x)$ is concave up (down) on I.

Theorem (Second Derivative Check)

Let I be an open interval, $a \in I$, and $f : I \to \mathbb{R}$ be s.t. (i) $f'(a) = 0$ ($(a, f(a))$ is stationary.) (ii) $f''(a) < 0$ ($f''(a) > 0$), then $(a, f(a))$ is a relative maximum (minimum).

Idea: If $(a, f(a))$ is stationary point, we have 4 possible cases:



Roughly speaking, $f''(a) < 0 \Rightarrow f$ is concave down around $x = a$. This rules out case 1,2,3, and so f attains maximum at $x = a$.

Example

Let $f(x) = x - 2\sin x$.

Example

$f(x) = x^3 - 3x^2 - 9x + 5$

$f'(x)$ gives the monotonicity (increasing/decreasing) of $f(x)$;
$f''(x)$ gives the convexity (concave up/down) of $f(x)$;

definition

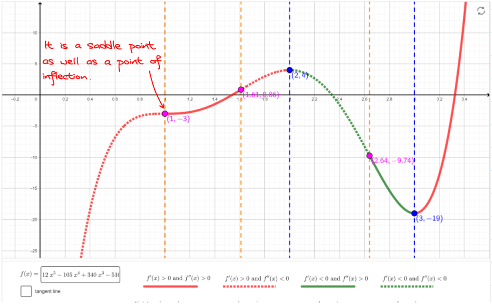
Let I be an open interval, $a \in I$ and $f : I \to \mathbb{R}$ be s.t.

- f is continuous.
- $f''(x) > 0$ ($f''(x) < 0$) for all $x \in I$ with $x < a$.
- $f''(x) < 0$ ($f''(x) > 0$) for all $x \in I$ with $x > a$.

then $(a, f(a))$ is a point of inflection of $f(x)$.

Exercise

Let $f(x) = x^2 e^x$. Find all extreme points and points of inflection of f.



Example

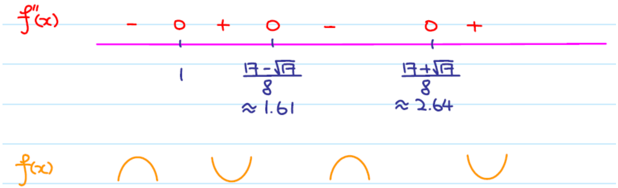
$f(x) = 12x^5 - 105x^4 + 340x^3 - 510x^2 + 360x - 120$
Find the range of x s.t. (1) $f'(x) > 0$, $f'(x) < 0$ (2) $f''(x) > 0$, $f''(x) < 0$

Step 1: Find $f'(x)$ and factorize it (using factor theorem)

$$\begin{aligned} f'(x) =& 60x^4 - 420x^3 + 1020x^2 - 1020x + 360 \\ =& 60(x^4 - 7x^3 + 17x^2 - 17x + 6) \\ =& 60(x-1)^2(x-2)(x-3) \end{aligned}$$

Step 2: gives intervals $x < 1, \quad 1 < x < 2, \quad 2 < x < 3, \quad x > 3.$

$$
\begin{aligned}
f''(x) =& 240x^3 - 1260x^2 + 2040x - 1020 \\
=& 60(x-1)(4x^2 - 17x + 17) \\
=& 240(x-1)\left[x - \left(\frac{17+\sqrt{17}}{8}\right)\right]\left[x - \left(\frac{17-\sqrt{17}}{8}\right)\right]
\end{aligned}
$$



Points of inflection: $(1, -23), \left(\frac{17\pm\sqrt{17}}{8}, f\left(\frac{17\pm\sqrt{17}}{8}\right)\right)$

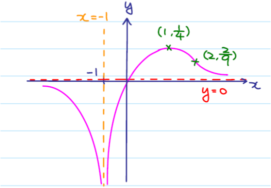
Example

$f(x) = \frac{x}{(x+1)^2}$, $x \neq -1$.

$$f'(x) = \frac{1-x}{(x+1)^3}, \quad f''(x) = \frac{2(x-2)}{(x+1)^4}$$

max at $(1, \frac{1}{4})$, point of inflection:$(2, \frac{2}{9})$

The graph of $y = f(x)$ behaves like:
the vertical line $x = -1$, when x is "near" -1.
the horizontal line $y = 0$, when x is "near" $\pm\infty$
In fact, $x = -1$ is called a vertical asymptote,
and $y = 0$ is called a horizontal asymptote.



Asymptotes

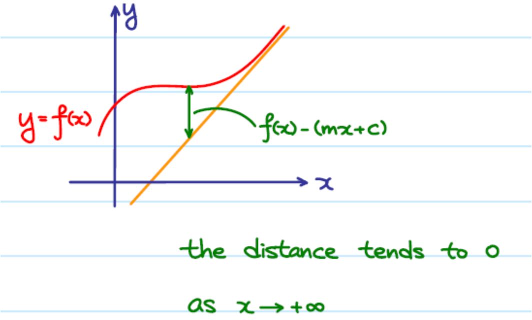
Definition

- If $\lim_{x\to a^+} f(x)$ or $\lim_{x\to a^-} f(x) = +\infty$ or $-\infty$, then $x = a$ is said to be a vertical asymptote.
- If $\lim_{x\to+\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$, with $L \in \mathbb{R}$, then $y = L$ is said to be a horizontal asymptote.
- If $y = mx + c$ is a line s.t.

$$\lim_{x\to+\infty} f(x) - (mx + c) = 0 \text{ or } \lim_{x\to-\infty} f(x) - (mx + c) = 0,$$

then it is said to be a slant asymptote (or oblique asymptote) of f.

Note: both $\lim_{x\to+\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ exist but differ.


y
y = f(x)
f(x) - (mx+c)
x
the distance tends to 0
as x → +∞

Example

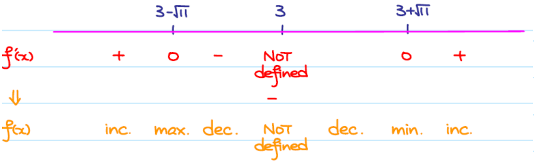
Let

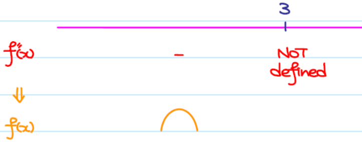
$$f(x) = \frac{x^2 + 3x - 7}{x - 3}, \qquad x \neq 3.$$

$$f'(x) = \frac{x^2 - 6x - 2}{(x-3)^2}, \qquad f''(x) = \frac{22}{(x-3)^3}$$

$$\max = (3-\sqrt{11}, f(3-\sqrt{11})) = (3-\sqrt{11}, 9-2\sqrt{11})$$
$$\min = (3+\sqrt{11}, f(3+\sqrt{11})) = (3+\sqrt{11}, 9+2\sqrt{11})$$

No point of inflection.





Vertical asymptote $x = 3$.

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} \frac{x^2+3x-7}{x-3} = -\infty,$$
$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} \frac{x^2+3x-7}{x-3} = +\infty$$

slant asymptote:

$$x^2+3x-7 = (x-3)(x+6)+11$$
$$f(x) = \frac{x^2+3x-7}{x-3} = x+6+\frac{11}{x-3}$$

with $x+6$ the slant asymptote.
Explanation: $\lim_{x\to\pm\infty} f(x)-(x+6) = \lim_{x\to\pm\infty} \frac{11}{x+3} = 0$

Using long division to find slant asymptote only works for the case that $f(x)$ is a rational function. Generally

$$m = \lim_{x \to +\infty} \frac{f(x)}{x}, \quad c = \lim_{x \to +\infty} f(x) - mx.$$

If anyone of them does NOT exist, there is no slant asymptote. If both limits exist, $y = mx + c$ is a slant asymptote at $+\infty$. Similarly, one can define slant asymptote at $-\infty$.

Sketch $y = f(x)$.

Step 1: draw asymptotes.
Step 2: put down x- & y-intercepts.

$\frac{x^2+3x-7}{x-3} = 0$, i.e., $x^2 + 3x - 7 = 0$

$x = \frac{-3 \pm \sqrt{37}}{2}$.

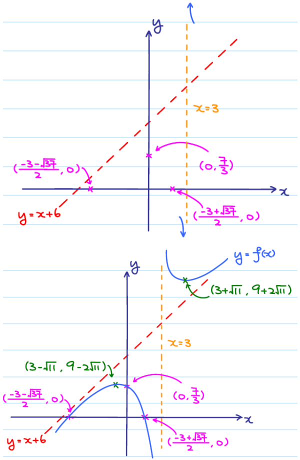
y-intercept: $f(0) = \frac{7}{3}$.

Step 3:

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} \frac{x^2+3x-7}{x-3} = -\infty$$

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} \frac{x^2+3x-7}{x-3} = +\infty$$

Step 4: Use the information $f'(x)$ and $f''(x)$.



Exercise 6.5.2

Let $f(x) = \frac{x^2+1}{(x-1)^2}$, for $x \neq 1$

- Find $f'(x)$ and all extreme points.
- Find $f''(x)$ and all points of inflection.
- Find all asymptotes of $f(x)$.
- Find x-intercept(s) and y-intercept of $f(x)$.
- Sketch the graph of $f(x)$.

curve sketching captures the main features of graph of $f(x)$

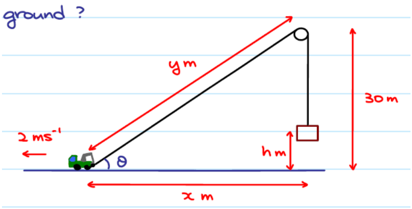
- x-intercept, i.e., solve $f(x) = 0$
- y-intercept, i.e., y-intercept $= f(0)$
- increang/decreasing saddle point/max./min, i.e. solve $f'(x) > 0/f'(x) < 0$ and change of sign of $f'(x)$?
- concave/convex point of inflection, i.e., solve $f''(x) > 0/f''(x) < 0$, change of sign of $f''(x)$?
- vertical asymptote: any $x = a$ with $\lim_{x\to a^+} f(x) = \pm\infty$ or $\lim_{x\to a^-} f(x) = \pm\infty$
- horizontal asymptote: $m = \lim_{x\to+\infty} f(x)$
- slant asymptote: $m = \lim_{x\to+\infty} \frac{f(x)}{x}$, $c = \lim_{x\to+\infty} f(x) - mx$

Rate of change

$\frac{dy}{dx}$ is the rate of change of y with respect to x.

Example

A weight is lifted by a rope which passes through a pulley. The other end of the rope is pulled by a truck that moves at 2 ms^{-1}. If the pulley is 30 m above the ground, how fast is the weight rising when the rope makes an angle $\frac{\pi}{6}$ with the ground?



- Given: $\frac{dx}{dt} = 2$
- Question: when $\theta = \frac{\pi}{6}$, $\frac{dh}{dt} = ?$
- Relations of variables:

$$x^2 + 30^2 = y^2, \quad x \tan\theta = 30, \quad \frac{dy}{dt} = \frac{dh}{dt}$$

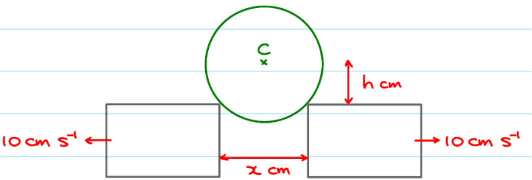
- When $\theta = \frac{\pi}{6}$, $x \tan\frac{\pi}{6} = 30 \Rightarrow x = 30\sqrt{3}$
- $(30\sqrt{3})^2 + 30^2 = y^2 \Rightarrow y = 60$
- Differentiate the first identity in t:

$$2x\frac{dx}{dt} = 2y\frac{dy}{dt}, \quad 2(30\sqrt{3})(2) = 2(60)\frac{dy}{dt}$$

Hence, $\frac{dh}{dt} = \frac{dy}{dt} = \sqrt{3}$

Example

A cylindrical block falls vertically thrusting two rectangular blocks apart with equal horizontal speed. When the center C of the cylinder is 20 cm above the level of the blocks, they are 30 cm apart and are moving at 10 cms^{-1}. Find the speed of the center of the cylinder.



- Question: When $x = 30$, $h = 20$, $\frac{dx}{dt} = 2(10) = 20$, $\frac{dh}{dt} = ?$
- Relations of variables: $(\frac{x}{2})^2 + h^2 = R^2$, where R is the radius of the cylinder.
- Differentiate in t: $\frac{1}{2}x\frac{dx}{dt} + 2h\frac{dh}{dt} = 0$
- When $x = 30$, $h = 20$, $\frac{dx}{dt} = 20$:

$$\frac{1}{2}(30)(20) + 2(20)\frac{dh}{dt} = 0, \quad \frac{dh}{dt} = -\frac{15}{2}$$

It is negative, which means the cylindrical block falls down.

Exercise

A metal cube is expending uniformly as it is heated. At time t (sec), the length of each edge of the cube is x cm, and the volume of the cube is V cm^3. Given that the volume of the cube increases at a constant rate of 0.048 $cm^3 s^{-1}$. When the length of one side of the cube is 8 cm, find

(a) the rate of increase of the length of a side of the cube; (Hint: When $x = 8$, $\frac{dx}{dt} =$?. Note that $V = x^3$, differentiate both sides in t by using the chain rue)

(b) the rate of increase of the total surface area of the cube. (Hint: total surface area $A = 6x^2$, differentiate both sides in t.)

Taylor theorem

Let $f(x)$ be a function with derivatives of all orders on I, and $c \in I$.
idea of Taylor theorem: approximate f around $x = c$ by a polynomial $T_n(x)$ of a degree n?

$$f^{(i)}(c) = T_n^{(i)}(c), \quad i = 0, 1, \ldots, n$$

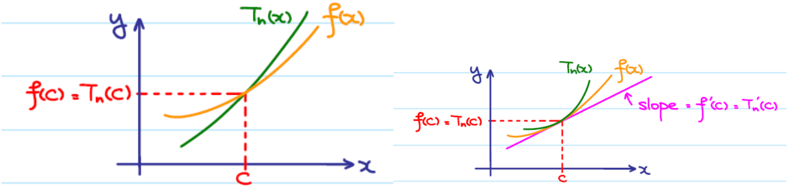
($n + 1$ conditions) $f(x)$ and $T_n(x)$ agree with each other up to the n-th derivative at $x = c$. Let

$$T_n(x) = d_0 + d_1(x - c) + d_2(x - c)^2 + \cdots + d_n(x - c)^n = \sum_{k=0}^{n} d_k(x - c)^k$$

$d_0, d_1, \cdots, d_n$ are constants to be determined.

$n + 1$ conditions, $n + 1$ constants $\Rightarrow d_k$'s are completely determined.

If $f(c) = T_n(c)$, the graphs of $f(x)$ and $T_n(x)$ intersect at $x = c$. Moreover, if $f'(c) = T_n'(c)$, the graphs of f and T_n share the tangent line at $x = c$.



$f(x)$ and $T_n(x)$ agree with each other up to nth derivative at $x = c$ is the generalization of the above. Hopefully, increasing n (the degree if $T_n(x)$) will give better approximation of $f(x)$ around $x = c$.

To determine d_k :

$$\begin{aligned}
T_n(x) =& d_0 + d_1(x-c) + d_2(x-c)^2 + \cdots + d_n(x-c)^n.\\
f(c) =& T_n(c) = d_0.\\
T_n'(x) =& d_1 + 2d_2(x-c) + 3d_3(x-c)^2 + \cdots + nd_n(x-c)^{n-1}.\\
f'(c) =& T_n'(c) = d_1 \Rightarrow d_1 = \frac{f'(c)}{1!}.\\
T_n''(x) =& 2\cdot 1\cdot d_2 + 3\cdot 2d_3(x-c) + \cdots + n(n-1)d_n(x-c)^{n-2}\\
f''(c) =& T_n''(c) = 2!d_2 \Rightarrow d_2 = \frac{f''(c)}{2!}.
\end{aligned}$$

Repeating the process, in general, we have

$$d_k = \frac{f^{(k)}(c)}{k!} \quad k = 0, 1, 2, \cdots, n.$$

Definition

Let I be an open interval and $c \in I$, $f : I \to \mathbb{R}$ has nth derivative at c.

$$\begin{aligned} T_n(x) &= f(c) + \frac{f'(c)}{1!}(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n \\ &= \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k \end{aligned}$$

is called the Talyor polynomial of order n generated by f at $x = c$.

$T_1(x) = f(c) + f'(c)(x-c)$ is in fact the linearization of $f(x)$ at $x = c$.

Example

Let $f(x) = e^x$. Find Taylor polynomial $T_n(x)$ of f at $x = 0$.

Example

Let $f(x) = e^x$. Find the Taylor polynomials $T_n(x)$ of f at $x = 2$.

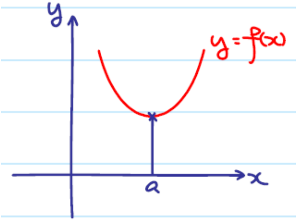
Example 7.1.3

Let $f(x) = \sqrt{x}$, find the Taylor polynomial $T_2(x)$ of degree 2 of f at $x = 100$, and approximate $\sqrt{101}$.

If $f'(a) = 0$ and $f''(a) > 0$, then when x is "close" to a,

$$f(x) \approx f(a) + \frac{f''(a)}{2}(x-a)^2$$

locally, like a parabola opening upward !
It suggests why f attains a local minimum at $x = a$.



How about $f''(a) = 0$?

Example

Let $f(x) = \cos x$.

Then (k is a non negative integer.)

$$f^{(m)}(x) = \begin{cases} \cos x & \text{if } m = 4k \\ -\sin x & \text{if } m = 4k+1 \\ -\cos x & \text{if } m = 4k+2 \\ \sin x & \text{if } m = 4k+3 \end{cases}$$

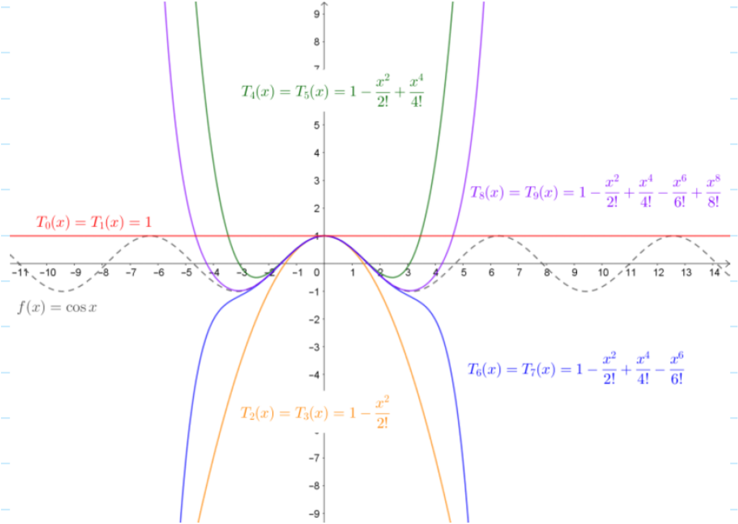
$$f^{(m)}(0) = \begin{cases} 1 & \text{if } m = 4k \\ 0 & \text{if } m = 4k+1 \\ -1 & \text{if } m = 4k+2 \\ 0 & \text{if } m = 4k+3 \end{cases}$$

Let $T_n(x)$ be the Taylor polynomial of degree n generated by f at $x = 0$.

$$
\begin{aligned}
T_0(x) &= f(0) = 1 \\
T_1(x) &= f(0) + f'(0)x = 1 \\
T_2(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 = 1 - \frac{x^2}{2!} \\
T_3(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 = 1 - \frac{x^2}{2!} \\
T_4(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 \\
&= 1 - \frac{x^2}{2!} + \frac{x^4}{4!}
\end{aligned}
$$

In general, for $n \geq 0$,

$$
\begin{aligned}
&T_{2n}(x) = T_{2n+1}(x) \\
=& f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(2n)}(0)}{(2n)!}x^{2n} + \frac{f^{(2n+1)}(0)}{(2n+1)}x^{2n+1} \\
=& 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + \frac{(-1)^n}{(2n)!}x^{2n} = \sum_{k=0}^{n} \frac{(-1)^k}{(2k)!}x^{2k}
\end{aligned}
$$


$T_4(x) = T_5(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$
$T_8(x) = T_9(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!}$
$T_0(x) = T_1(x) = 1$
$f(x) = \cos x$
$T_6(x) = T_7(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$
$T_2(x) = T_3(x) = 1 - \frac{x^2}{2!}$

Example

Let $f(x) = \sin x$.

Then (k a non negative integer)

$$f^{(m)}(x) = \begin{cases} \sin x & \text{if } m = 4k \\ \cos x & \text{if } m = 4k+1 \\ -\sin x & \text{if } m = 4k+2 \\ -\cos x & \text{if } m = 4k+3 \end{cases} \qquad f^{(m)}(0) = \begin{cases} 0 & \text{if } m = 4k \\ 1 & \text{if } m = 4k+1 \\ 0 & \text{if } m = 4k+2 \\ -1 & \text{if } m = 4k+3 \end{cases}$$

Let $T_n(x)$ be the Taylor polynomial of degree n of f at $x = 0$.

$$T_0(x) = f(0) = 0$$

$$T_1(x) = f(0) + f'(0)x = x$$

$$T_2(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 = x$$

$$T_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 = x - \frac{x^3}{6}$$

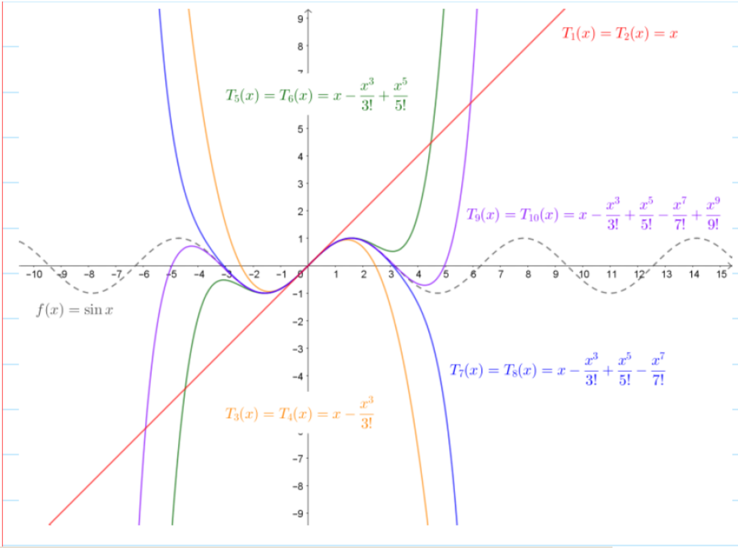
$$T_4(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 = x - \frac{x^3}{6}$$

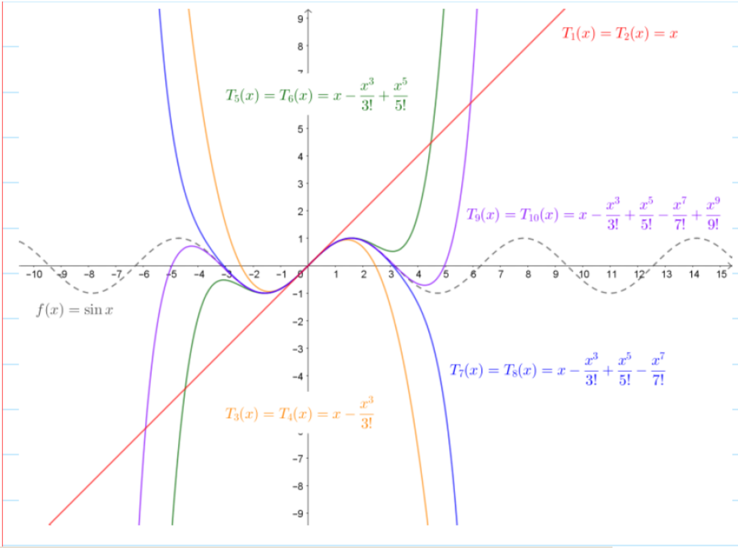
In general, for $n \geq 0$,

$$T_{2n+1}(x) = T_{2n+2}(x)$$

$$= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(2n+1)}(0)}{(2n+1)!}x^{2n+1} + \frac{f^{(2n+2)}(0)}{(2n+2)!}x^{2n+2}$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{(-1)^n}{(2n+1)!}x^{2n+1} = \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)!}x^{2k+1}$$


$T_1(x) = T_2(x) = x$
$T_5(x) = T_6(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$
$T_9(x) = T_{10}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!}$
$f(x) = \sin x$
$T_7(x) = T_8(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$
$T_3(x) = T_4(x) = x - \frac{x^3}{3!}$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$f(x) = \sin x$
$T_7(x) = T_8(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$
$T_3(x) = T_4(x) = x - \frac{x^3}{3!}$

Taylor's Theorem and Taylor Series

Question

Let $T_n(x)$ be the Taylor polynomial of degree n of $f(x)$ at $x = c$. When approximating $f(x)$ by $T_n(x)$, how good is the approximation?

Taylor's theorem

If f and the derivatives $f', f'', \cdots, f^{(n)}$ are continuous on the closed interval between x and c, $f^{(n)}$ is differential on the open interval between x and c, then there exists η between x and c s.t.

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k + \frac{f^{(n+1)}(\eta)}{(n+1)!}(x-c)^{n+1}$$
$$:= T_n(x) + R_n(x).$$

When approximating $f(x)$ by $T_n(x)$, the error is of the form

$$R_n(x) = \frac{f^{(n+1)}(\eta)}{(n+1)!}(x-c)^{n+1}.$$

That is the error can be described by $f^{(n+1)}$

Fix $x \in \mathbb{R}$, and assume $x > c$. Let

$$F(t) = f(t) - T_n(t) - \frac{f(x) - T_n(x)}{(x-c)^{n+1}}(t-c)^{n+1}$$

Then F is continuous on $[c, x]$, diff. on (c, x), $F(c) = F(x) = 0$.
By Rolle's theorem, $\exists \eta_1 \in (c, x)$ s.t. $F'(\eta_1) = 0$

$$F'(t) = f'(t) - T_n'(t) - (n+1)\frac{f(x) - T_n(x)}{(x-c)^{n+1}}(t-c)^{n}$$

Then F' is continuous on $[c, \eta_1]$, diff. on (c, η_1), $F'(c) = F'(\eta_1) = 0$
By Rolle's theorem, $\exists \eta_2 \in (c, \eta_1)$ such that $F''(\eta_2) = 0$.

$$F''(t) = f''(t) - T_n''(t) - (n+1)n\frac{f(x) - T_n(x)}{(x-c)^{n+1}}(t-c)^{n-1}$$

Then F'' is continuous on $[c, \eta_2]$, diff. on (c, η_2), $F''(c) = F''(\eta_2) = 0$
By Rolle's theorem, $\exists \eta_3 \in (c, \eta_2)$ such that $F'''(\eta_2) = 0$.

Repeating the process :$\exists \eta_{n+1} \in (c, \eta_n)$ such that $F^{(n+1)}(\eta_{n+1}) = 0$.

$$F^{(n+1)}(t) = f^{(n+1)}(t) - (n+1)!\frac{f(x) - T_n(x)}{(x-c)^{n+1}}$$

with $(T_n^{(n+1)}(t) = 0)$

$$0 = F^{(n+1)}(\eta_{n+1}) = f^{(n+1)}(\eta_{n+1}) - (n+1)!\frac{f(x) - T_n(x)}{(x-c)^{n+1}}$$

Thus

$$f(x) = T_n(x) + \frac{f^{(n+1)}(\eta)}{(n+1)!}(x-c)^{n+1}$$

by letting $\eta = \eta_{n+1} \in (c, x)$. The proof for the case $x < c$ is similar.

Example

Approximate $\cos 0.1$.

$$T_5(x) = T_4(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$$

Taylor polynomials at $x = 0$.

$$\cos 0.1 \approx T_5(0.1) = 0.995004166\cdots$$

By Taylor's Theorem (put $n = 5$, $c = 0$ and $x = 0.1$), $\exists \eta \in (0, 0.1)$ s.t.

$$\cos(0.1) = T_5(0.1) + \frac{\cos^{(6)}(\eta)}{6!}(0.1)^6$$

the error

$$\Big|\cos(0.1) - T_5(0.1)\Big| = |\frac{f^{(6)}(\eta)}{6!}(0.1)^6| \leq \frac{1}{6!}(0.1)^6 \approx 1.38 \times 10^{-9}.$$

Example

Let $f(x) = \sqrt{x}$.

Taylor polynomial $T_2(x)$ of degree 2 at $x = 100$ is

$$T_2(x) = f(100) + f'(100)(x - 100) + \frac{f''(100)}{2!}(x - 100)^2$$
$$= 10 + \frac{1}{20}(x - 100) - \frac{1}{8000}(x - 100)^2$$

approximate $\sqrt{101} = f(101)$ by

$$\sqrt{101} = f(101) \approx T_2(101) = 10 + \frac{1}{20} - \frac{1}{8000} = 10.049875$$

By Taylor's Theorem (put $n = 2$, $c = 100$ and $x = 101$), there exists $\eta \in (100, 101)$ such that

$$f(101) = T_2(101) + \frac{f'''(\eta)}{3!}(101 - 100)^3,$$

with the error

$$= \left| f(101) - T_2(101) \right| = \left| \frac{f'''(\eta)}{3!}(101 - 100)^3 \right|$$

$$< \frac{1}{3!}\left(\frac{3}{8(\sqrt{\eta})^5}\right) < \frac{1}{3!}\left(\frac{3}{8} \times \frac{1}{10^5}\right) = 6.25 \times 10^{-7}$$

Observation: if $T_n(x)$ is Taylor polynomial of degree n of f at $x = c$, then $T_n(x)$ gives better approximation around $x = c$ as n increases.

Definition

Let $f^{(n)}(c)$ exist for all $n = 0, 1, 2 \cdots$. Taylor series generated by f at $x = c$ is defined by

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c}{n!}(x-c)^n = f(c) + f'(c)(x-c) + \frac{f^{(n)}(c)}{n!}(x-c)^n + \cdots$$

In particular, if $c = 0$, the series is called Maclaurin series.

Note: Taylor series of f at $x = c$ is a power series centered at $x = c$.

Example

Let $f(x) = \frac{1}{1-x}$. Find Taylor series of f

In fact, $T(x)$, converges to $f(x)$ for $-1 < x < 1$.

$$T_n(0.1) = 1 + 0.1 + \cdots + 0.1^n = 1.111\cdots 1 \approx f(0.1) = \frac{1}{1-0.1} = 1.111$$
$$T_n(2) = 1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1 \not\approx f(2) = \frac{1}{1-2} = -1.$$

$T_n(x)$ gets closer to $f(x)$ for$-1 < x < 1$ as $n \to \infty$, and $f(x) = T(x)$ for $-1 < x < 1$

Example

Let

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

$$f(0) = f'(0) = f''(0) = \cdots = 0$$

the Taylor series of f at $x = 0$ is

$$T_n(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = 0$$

converges for all $x \in \mathbb{R}$. However, $0 = T(x) \neq f(x) = e^{-\frac{1}{x^2}}$ for all $x = 0$

Technical issues

- Convergence of Taylor series ?
- Does $T(x)$ converge to $f(x)$?

key idea:

$$f(x) = T_n(x) + R_n(x), \quad R_n(x) = f(x) - T_n(x)$$

If $\lim_{n\to\infty} R_n(x) = 0$, that is error tends to 0, then

$$f(x) = \lim_{n\to\infty} T_n(x) = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k = \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k$$

$$= f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \cdots$$

Theorem

Let I be an interval, c an interior point of I, and let $f : I \to \mathbb{R}$. If $f^{(n)}(c)$ exists for $n \in \mathbb{Z}^+$, $T_n(x)$ is the Taylor polynomial of degree n of f at $x = c$ and $R_n(x) = f(x) - T_n(x)$. If $\lim_{n\to\infty} R_n(x) = 0$, for all $x \in I$, the Taylor series

$$T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n \text{ converges to } f(x) \text{ for all } x \in I, \text{ i.e.}$$

$$\begin{aligned} f(x) &= f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \cdots \\ &= \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n \quad \forall x \in I. \end{aligned}$$

Example

Let $f(x) = \cos x$

$$f(x) = \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + R_{2n+1}(x)$$

Frequently used Taylor series (MacLaurin series)

- $\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots = \sum_{n=0}^{\infty} x^n, \forall |x| < 1$
- $\frac{1}{1+x} = 1 - x + x^2 - \cdots + (-)x^n + \cdots = \sum_{n=0}^{\infty} (-1)^n x^n, \forall |x| < 1$
- $e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}\ \forall x \in \mathbb{R}$
- $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$, $\forall x \in \mathbb{R}$
- $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}, \forall x \in \mathbb{R}$
- $\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n+1} \frac{x^n}{n} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n$, $\forall -1 < x \leq 1$

Operations on Taylor series: from frequently used Taylor series, we can find the Taylor series of more complicated function

- (Addition) $\cos x + \sin x = 1 + x - \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots$
- (Subtraction) $\cos x - \sin x = 1 - x - \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$
- (Product)
- (Composition)

$$e^{\sin x} = 1 + (x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots) + \frac{1}{2!}(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots)^2 + \frac{1}{3!}(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots)^3 + \cdots = 1 + x + \frac{x^2}{2} + \cdots$$

- division
- differentiation / (integration)

Example

Find the Taylor series of $f(x) = \frac{8x-7}{2x^2-5x+2}$ at $x = 0$

$$f(x) = \frac{8x-7}{2x^2-5x+2} = \frac{3}{2-x} + \frac{2}{1-2x}$$

Example

Find the Taylor series of $f(x) = \frac{1}{5-x}$ at $x = 2$.

Example

Find the Taylor series of $\frac{1}{(1+x)^2}$ at $x = 0$ using $\frac{d}{dx}\frac{1}{1+x} = -\frac{1}{(1+x)^2}$.

Exercise

Given the Taylor series of $\sin x$ at $x = 0$, using $\frac{d}{dx}\sin x = \cos x$, find the Taylor series of $\cos x$ at $x = 0$.

Example

Find the Taylor series generated by $\tan^{-1}(x)$ at $x = 0$ using $\int \frac{1}{1+x^2} dx = \tan^{-1} x + c$.

Exercise

Pretend we only know the Taylor series of $\frac{1}{1-x}$ at $x = 0$

- By replacing x by $-x$, find the Taylor series of $\frac{1}{1+x}$ at $x = 0$
- By considering $\int \frac{1}{1+x} dx = \ln(1+x) + c$ for $x > -1$, find the Taylor series of $\ln(1+x)$ at $x = 0$

summary

- convexity (concave up / down)
- second derivative check
- asymptote (horizontal, vertical, slant)
- Taylor series (of common functions)