11. Let V and W be finite-dimensional vector spaces over F, and let ψ_1 and ψ_2 be the isomorphisms between V and V** and W and W**, respectively, as defined in Theorem 2.26. Let $\mathsf{T}\colon \mathsf{V} \to \mathsf{W}$ be linear, and define $\mathsf{T}^{tt} = (\mathsf{T}^t)^t$. Prove that the diagram depicted in Figure 2.6 commutes (i.e., prove that $\psi_2\mathsf{T} = \mathsf{T}^{tt}\psi_1$).

dual space

dual map / transpose

T^t is T^*

different notation

$$\begin{tikzcd}
v \in \mathsf{V} \arrow[r, "\mathsf{T}"] \arrow[d, "\psi_1"'] & \mathsf{W} \ni w \arrow[d, "\psi_2"] \\
\psi_1(v) = \hat{v} \in \mathsf{V}^{**} \arrow[r, "\mathsf{T}^{tt}"] & \mathsf{W}^{**} \ni \psi_2(w) = \hat{w}
\end{tikzcd}$$

Figure 2.6

- Recall that V is isomorphic to V^{**}

$$\psi : V \to V^{**} \quad v \mapsto \varphi(v) \qquad \text{where} \quad \varphi(v) : V^* \to F \quad f \mapsto \varphi(v)(f) := f(v)$$

ψ is isomorphism

So $\begin{cases} \text{let } \hat{v} \text{ denotes } \psi_1(v) \text{ for any } v \in V \\ \text{let } \hat{w} \text{ denotes } \psi_2(w) \text{ for any } w \in W \end{cases}$

- Let $\begin{cases} \beta = \{v_1 \cdots v_n\} \text{ be basis for } V \\ \gamma = \{w_1 \cdots w_m\} \text{ be basis for } W, \end{cases}$ and $\begin{cases} \beta^* = \{f_1 \cdots f_n\} \text{ dual basis of } \beta \\ \gamma^* = \{g_1 \cdots g_m\} \text{ dual basis of } \gamma \end{cases}$

Consider $\begin{cases} \beta^{**} = \{\hat{v}_1 \; \cdots \; \hat{v}_n\} \\ \gamma^{**} = \{\hat{w}_1 \; \cdots \; \hat{w}_m\} \end{cases}$ $\quad \begin{matrix} \hat{v}_i = \varphi_1(v_i) & i = 1 \cdots n \\ \hat{w}_j = \varphi_2(w_j) & j = 1 \cdots m \end{matrix}$

Then $\begin{cases} \hat{v}_i(f_j) = f_j(v_i) = \delta_{ij} \Rightarrow \beta^{**} \text{ is dual basis of } \beta^* \\ \hat{w}_i(g_j) = g_j(w_i) = \delta_{ij} \Rightarrow \gamma^{**} \text{ is dual basis of } \gamma^* \end{cases}$

- $T : V \to W \quad v \mapsto w$ $\qquad T^* : W^* \to V^* \quad g \mapsto g \circ T$ $\qquad T^{**} : V^{**} \to W^{**} \quad \hat{v} \mapsto \hat{v} \circ T^*$

$$W^* \xrightarrow{T^*} V^* \xrightarrow{\hat{v}} F$$

- Let $A = [T]_\beta^\gamma$

- For $j = 1 \cdots n$

$$\begin{cases} \psi_2 \circ T(v_j) = \psi_2\left(\sum_{i=1}^m A_{ij} \cdot w_i\right) = \sum_{i=1}^m A_{ij}\, \psi_2(w_i) = \sum_{i=1}^m A_{ij}\, \hat{w}_i \\ T^{**} \circ \psi_1(v_j) = T^{**}(\hat{v}_j) = \hat{v}_j \circ T^* \end{cases}$$

Claim: $\sum_{i=1}^{m} A_{ij} \cdot \hat{w}_i = \hat{v}_j \circ T^* \in W^{**}$ $\quad W^* \to F$

To prove that they agree on β^*. basis for W^*

for $k = 1 \cdots m$

$$\begin{cases} \psi_2 \circ T(v_j)(g_k) = \sum_{i=1}^{m} A_{ij} \hat{w}_i (g_k) = \sum_{i=1}^{m} A_{ij} \delta_{ik} = A_{kj} \\ T^{**} \circ \psi_1 (v_j)(g_k) = (\hat{v}_j \circ T^*)(g_k) = \hat{v}_j (T^*(g_k)) \end{cases}$$

$$= \hat{v}_j (g_k \circ T) = g_k \circ T (v_j)$$

$$= g_k (T(v_j)) = g_k \left(\sum_{i=1}^{m} A_{ij} w_i \right)$$

$$= \sum_{i=1}^{m} A_{ij} \cdot g_k(w_i) = \sum_{i=1}^{m} A_{ij} \delta_{ki} = A_{kj}$$

#

Sec 5.1 Q17

17. Let T be the linear operator on $\mathsf{M}_{n\times n}(R)$ defined by $\mathsf{T}(A) = A^t$.

(a) Show that ± 1 are the only eigenvalues of T.

(b) Describe the eigenvectors corresponding to each eigenvalue of T.

(c) Find an ordered basis β for $\mathsf{M}_{2\times 2}(R)$ such that $[\mathsf{T}]_\beta$ is a diagonal matrix.

(d) Find an ordered basis β for $\mathsf{M}_{n\times n}(R)$ such that $[\mathsf{T}]_\beta$ is a diagonal matrix for $n > 2$.

(a) $A^t = T(A) = \lambda A \quad \therefore \quad A = (A^t)^t = T^2(A) = \lambda^2 A \qquad A \neq 0_{2\times 2}$

$\therefore \lambda^2 = 1 \Rightarrow \lambda = \pm 1$

(b) if $\lambda = 1 \quad A^t = T(A) = \lambda A = A \Rightarrow A$ is symmetric

if $\lambda = -1 \quad A^t = T(A) = \lambda A = -A \Rightarrow A$ is skew-symmetric

(c) let $\beta' = \{M_{11}, M_{12}, M_{21}, M_{22}\}$ basis for $M_{2\times 2}$

$$[T]_{\beta'} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad f_T(t) = \det([T]_{\beta'} - tI_4) = (t-1)^3(t+1)$$

eigen values are $\lambda_1 = 1$ and $\lambda_2 = -1$

For $\lambda_1 = 1$. $\quad B_1 = [T]_{\beta'} - \lambda_1 I_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$$B_1 x = \vec{0} \Leftrightarrow \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow x_2 = x_3 \Leftrightarrow x \in \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_4 \end{pmatrix} \in \mathbb{R}^4 : x_2 = x_3 \right\}$$

$$\therefore E_{\lambda_1} = \{ a_1 M_{11} + a_2(M_{12} + M_{21}) + a_3 M_{22} : a_i \in \mathbb{R} \}$$
$$= \text{span}(\{ M_{11}, M_{12} + M_{21}, M_{22} \})$$

For $\lambda_2 = -1$. $\quad B_2 = [T]_{\beta'} - \lambda_2 I_4 = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

$$B_2 x = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{cases} x_1 = 0 \\ x_2 = -x_3 \\ x_4 = 0 \end{cases} \Leftrightarrow x \in \{ x \in \mathbb{R}^4 : x_2 + x_3 = 0,\ x_1 = x_4 = 0 \}$$

$$\therefore E_{\lambda_2} = \{ a(M_{12} - M_{21}) : a \in \mathbb{R} \} = \text{span}(\{ M_{12} - M_{21} \})$$

$\beta = \{ M_{11}, M_{12} + M_{21}, M_{22}, M_{12} - M_{21} \}$ is an basis for $M_{2\times 2}$

s.t $\quad [T]_\beta = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{bmatrix}$

(d) $\beta = \{ M_{ii} : i = 1, \ldots, n \} \cup \{ M_{ij} + M_{ji} : 1 \le i < j \le n \} \cup \{ M_{ij} - M_{ji} : 1 \le i < j \le n \}$

3. Let A be an $n \times n$ matrix with characteristic polynomial

$$f(t) = (-1)^n t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0$$

(a) Prove that $f(0) = a_0 = \det(A)$. Deduce that A is invertible if and only if $a_0 \neq 0$. trivial.

(b) Prove that $f(t) = (A_{11}-t)(A_{22}-t)\cdots(A_{nn}-t)+q(t)$, where $q(t)$ is a polynomial of degree at most $n-2$. (Hint: Apply mathematical induction to n.)

(c) Show that $tr(A) = (-1)^{n-1}a_{n-1}$.

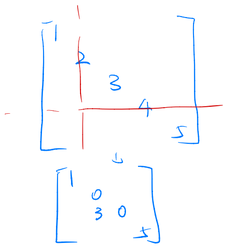
(b) · For $n=2$

$$\det(A-tI_2) = \det\left(\begin{bmatrix} A_{11}-t & A_{12} \\ A_{21} & A_{22}-t \end{bmatrix}\right) = (A_{11}-t)(A_{22}-t) - A_{12}A_{21}$$

$q(t) = -A_{12}A_{21}$ is of degree $0 = 2-2 = n-2$

· Suppose the case of $n-1$ holds.

· For $A \in M_{n\times n}$



$$\det(A-tI_n) = \det\left(\begin{bmatrix} A_{11}-t & & A_{1n} \\ & \ddots & \\ A_{n1} & & A_{nn}-t \end{bmatrix}\right)$$

$$= (A_{nn}-t)\cdot\det\left(\begin{bmatrix} A_{11}-t & & A_{1,n-1} \\ & \ddots & \\ A_{n-1,1} & & A_{n-1,n-1}-t \end{bmatrix}\right)$$

$$+ \sum_{j=1}^{n-1} A_{nj}(-1)^{n+j}\underbrace{\det\left(\begin{bmatrix} A_{11}-t & A_{12} & \cdots & A_{1j-1} & A_{1j+1} \cdots & A_{1,n-1} \\ A_{21} & A_{22}-t & \cdots & A_{2j-1} & A_{2j+1} \cdots & A_{2,n-1} \\ \vdots & \vdots & & & & \\ A_{n-1,1} & A_{n-1,2} & \cdots & A_{n-1,j-1} & A_{n-1,j+1} \cdots & A_{n-1,n}-t \end{bmatrix}\right)}$$

$n-1 \times n-1$

only $n-2$ t's in different row and col.

$$= (A_{nn}-t)\,q_n(t) + \sum_{j=1}^{n-1} A_{nj}\,q_j(t)$$

Obviously, $q_j(t) \in P_{n-2}$ and

by assumption, $q_n(t) = (A_{11}-t)\cdots(A_{n-1,n-1}-t) + q_n'(t)$ where $q_n'(t) \in P_{n-3}$

$$\therefore \det(A-tI_n) = (A_{11}-t)\cdots(A_{nn}-t) + \underbrace{(A_{nn}-t)\,q_n'(t) + \sum_{j=1}^{n-1} A_{nj}\,q_j(t)}_{\in P_{n-2}}$$

(c)

$$f(t) = (A_{11}-t)\cdots(A_{nn}-t) + \underbrace{q(t)}_{\in P_{n-2}}$$

$$= (-1)^n t^n + (-1)^{n-1}(A_{11}+\cdots+A_{nn})t^{n-1} + \underbrace{p(t)}_{\in P_{n-2}} + \underbrace{q(t)}_{\in P_{n-2}}$$

$$\therefore a_{n-1} = (-1)^{n-1}(A_{11}+\cdots+A_{nn}) = (-1)^{n-1}tr(A)$$