1. Given $\beta = \{1, 1+x, 1+x^2, 1+x^3\}$ a basis for $P_3(\mathbb{R})$
and $\gamma = \{e_{11}\ e_{12}\ e_{21}\ e_{22}\}$. the standard basis for $M_{2\times 2}(\mathbb{R})$

Let $T: P_3(\mathbb{R}) \to M_{2\times2}(\mathbb{R})$ be defined by
$$T(f(x)) = \begin{pmatrix} f'(0) & 2f(1) \\ 0 & f''(1) \end{pmatrix}$$
"linear"

Find $[T]_\beta^\gamma$, $N(T)$

Solution.
$$\begin{cases} T(1) = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} = 0\cdot e_{11} + 2\cdot e_{12} + 0\cdot e_{21} + 0\cdot e_{22} \\ T(1+x) = \begin{pmatrix} 1 & 4 \\ 0 & 0 \end{pmatrix} = 1\cdot e_{11} + 4\cdot e_{12} + 0\cdot e_{21} + 0\cdot e_{22} \\ T(1+x^2) = \begin{pmatrix} 0 & 4 \\ 0 & 2 \end{pmatrix} = 0\cdot e_{11} + 4\, e_{12} + 0\cdot e_{21} + 2\cdot e_{22} \\ T(1+x^3) = \begin{pmatrix} 0 & 4 \\ 0 & 6 \end{pmatrix} = 0\cdot e_{11} + 4\, e_{12} + 0\cdot e_{21} + 6\cdot e_{22} \end{cases}$$

$$[T]_\beta^\gamma = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 2 & 4 & 4 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 6 \end{bmatrix}$$

$$N(T) = \{ f \in P_3(\mathbb{R}) : T(f) = 0 \}$$
$$= \{ a_0 + a_1x + a_2x^2 + a_3x^3 : T(a_0 + a_1x + a_2x^2 + a_3x^3) = 0 \}$$

$$0 = T(a_0 + a_1x + a_2x^2 + a_3x^3)$$
$$= T\Big((a_0 - a_1 - a_2 - a_3)\cdot 1 + a_1(1+x) + a_2(1+x^2) + a_3(1+x^3) \Big)$$
$$= (a_0 - a_1 - a_2 - a_3)\, T(1) + a_1\cdot T(1+x) + a_2 \cdot T(1+x^2) + a_3 \cdot T(1+x^3)$$
$$= \begin{pmatrix} 0 & 2(a_0 - a_1 - a_2 - a_3) \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} a_1 & 4a_1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 4a_2 \\ 0 & 2a_2 \end{pmatrix} + \begin{pmatrix} 0 & 4a_3 \\ 0 & 6a_3 \end{pmatrix}$$
$$= \begin{pmatrix} a_1 & 2(a_0 + a_1 + a_2 + a_3) \\ 0 & 2a_2 + 6a_3 \end{pmatrix}$$

So $\begin{cases} a_1 = 0 \\ a_0 + a_1 + a_2 + a_3 = 0 \\ a_2 + 3a_3 = 0 \end{cases}$ let $a_3 = t$ be the arbitrary variable

Then $\begin{cases} a_0 = 2t \\ a_1 = 0 \\ a_2 = -3t \\ a_3 = t \end{cases}$

$$N(T) = \{ t\cdot(2 + 0\cdot x - 3\cdot x^2 + x^3) : t \in \mathbb{R} \}$$
$$\dim(N(T)) = 1$$

Sec 2.2 Q16

$<\infty$

16. Let V and W be vector spaces such that dim(V) = dim(W), and let T: V → W be linear. Show that there exist ordered bases β and γ for V and W, respectively, such that $[T]_\beta^\gamma$ is a diagonal matrix.

Proof: we first choose a basis $\{u_1, \cdots, u_k\}$ for $N(T)$ and extend the basis to a basis $\beta = \{u_1, \cdots, u_k, u_{k+1}, \cdots, u_n\}$ for V.

By dim thm, $\dim(V) = \dim(N(T)) + \dim(R(T))$

$$\dim(R(T)) = \dim(V) - \dim(N(T)) = n-k$$

Besides $R(T) = \text{span}(\{T(u_1), T(u_2), \cdots, T(u_n)\})$ $\quad T(u_1) = \cdots = T(u_k) = \vec{0}$

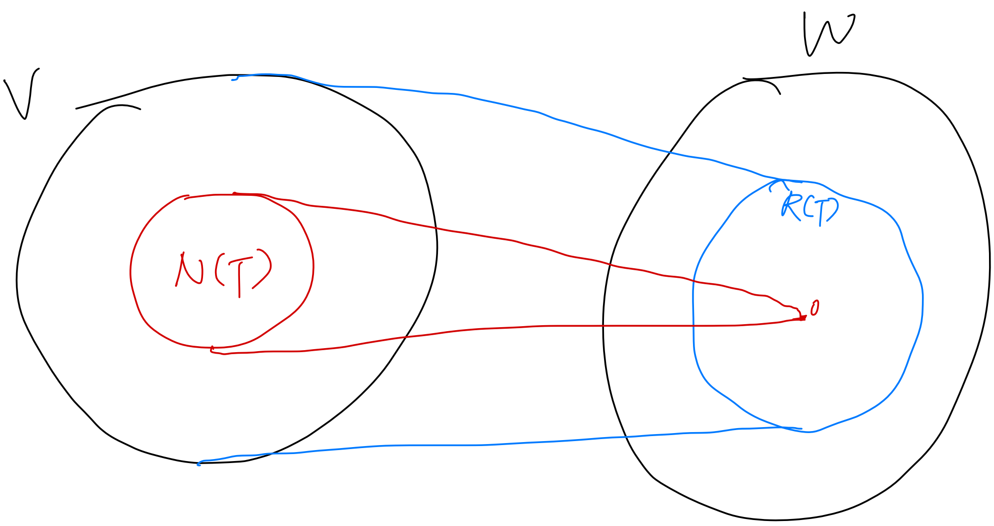
$$= \text{span}(\{T(u_{k+1}), \cdots, T(u_n)\})$$

Therefore, $\{T(u_{k+1}), T(u_{k+2}), \cdots, T(u_n)\}$ is a basis for $R(T)$

we extend it to a basis $\gamma = \{v_1, \cdots, v_k, T(u_{k+1}), \cdots, T(u_n)\}$ for W

Then.

$$[T]_\beta^\gamma = \begin{bmatrix} O_{k\times k} & O_{k\times n-k} \\ O_{n-k\times k} & I_{n-k} \end{bmatrix}$$ is a diagonal matrix



3. Sec 2.3. Q16

16. Let V be a finite-dimensional vector space, and let $\mathsf{T}\colon \mathsf{V} \to \mathsf{V}$ be linear.

(a) If $\operatorname{rank}(\mathsf{T}) = \operatorname{rank}(\mathsf{T}^2)$, prove that $\mathsf{R}(\mathsf{T}) \cap \mathsf{N}(\mathsf{T}) = \{0\}$. Deduce that $\mathsf{V} = \mathsf{R}(\mathsf{T}) \oplus \mathsf{N}(\mathsf{T})$ (see the exercises of Section 1.3).

(b) Prove that $\mathsf{V} = \mathsf{R}(\mathsf{T}^k) \oplus \mathsf{N}(\mathsf{T}^k)$ for some positive integer k.

(a) By rank-nullity thm, we have

$$\text{nullity}(T) = \dim(V) - \text{rank}(T) = \dim - \text{rank}(T^2) = \text{nullity}(T^2)$$

Since $N(T) \subset N(T^2)$, we have that $N(T) = N(T^2)$

Now, for any $v \in R(T) \cap N(T)$, $\exists\, u \in V$ s.t. $v = T(u)$

then $T^2(u) = T(T(u)) = T(v) = 0$, thus $u \in N(T^2)$

Since $N(T^2) = N(T)$, we have $v = T(u) = 0$

Thus $R(T) \cap N(T) = \{0\}$

Besides, $\dim(R(T) + N(T)) = \dim(R(T)) + \dim(N(T)) - \dim(R(T) \cap N(T))$

$$= \dim(V) - 0$$

Thus $R(T) + N(T) = V$, so $V = R(T) \oplus N(T)$

(b) for any $m \geq 0$. $R(T^{m+1}) \subset R(T^m)$

so $\text{rank}(T^m) \geq \text{rank}(T^{m+1})$

$\text{rank}(T) \geq \text{rank}(T^2) \geq \cdots \geq \text{rank}(T^k) \geq \cdots \geq 0$

since $\text{rank}(T) < \infty$, then only finite "$\geq$" above can be strict

So there exist k, such that $\text{rank}(T^k) = \text{rank}(T^{k+j})$ for any $j \geq 0$

Thus $\text{rank}(T^k) = \text{rank}(T^{k+k}) = \text{rank}((T^k)^2)$

By (a), we know $V = R(T^k) \oplus N(T^k)$

4. (Extension to Sec. 2.1: Q18) Please find **ALL** linear transformations $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $N(T) = R(T)$.

- If $N(T) = R(T)$ $\qquad \dim(R(T)) = \dim(N(T)) = 1$ $\Rightarrow T \neq T_0$

$\forall \vec{x} \in \mathbb{R}^2 \quad T^2(\vec{x}) = T(\underbrace{T(\vec{x})}_{\uparrow R(T) = N(T)}) = \vec{0} \in \mathbb{R}^2 \quad \Rightarrow T^2 = T_0$

Claim: $N(T) = R(T) \iff T^2 = T_0$ and $T \neq T_0$

- If $T^2 = T_0$, $T \neq T_0$

$\forall \vec{y} \in R(T) \quad \exists \vec{x} \in \mathbb{R}^2$ s.t. $\vec{y} = T(\vec{x})$

$T(\vec{y}) = T(T(\vec{x})) = T^2(\vec{x}) = T_0(\vec{x}) = \vec{0}$

Thus $\vec{y} \in N(T)$ and $R(T) \subset N(T)$

Besides $\begin{cases} T \neq T_0 \Rightarrow \dim(R(T)) \geq 1 \\ T^2 = T_0 \Rightarrow \dim(R(T)) \leq 1 \end{cases}$ $\Rightarrow \dim(R(T)) = 1$

$\Rightarrow \dim(N(T)) = 2 - 1 = 1$

Thus $R(T) = N(T)$

To find the concrete form of T,
we use the fact:

T is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$

$\iff T(\vec{x}) = A \cdot \vec{x}$, where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{R}^{2\times 2}$

Recall that $N(T) = R(T) \iff T^2 = T_0$ and $T \neq T_0$.

$T^2(\vec{x}) = A^2\vec{x}$ where $A^2 = \begin{pmatrix} a^2+bc & ab+bd \\ ac+cd & d^2+bc \end{pmatrix}$

$T^2 = T_0 \iff A^2 = O_{2\times2} \iff \begin{cases} a^2+bc = 0 \\ ab+bd = 0 \\ ac+cd = 0 \\ d^2+bc = 0 \end{cases}$

$T \neq T_0 \iff A \neq O_{2\times2}$

Case 1: $b=0 \Rightarrow a=d=0$

so $A = \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix}$ where $c \neq 0$.

Case 2: $b \neq 0 \Rightarrow a=-d$.

Besides, $(\det A)^2 = \det A^2 = \det O_{2\times2} = 0$

So $ad-bc = \det A = 0$, $c = -a^2/b$

Thus, $A = \begin{pmatrix} a & b \\ -a^2/b & -a \end{pmatrix}$ where $b \neq 0$, $a \in \mathbb{R}$