Lecture 10:

Change of coordinates

Prop: Let β and β' be two ordered bases for a finite-dim. vector space V, and let $Q = [I_V]_{\beta'}^{\beta}$.

$$V_{\beta'} \xrightarrow{I_V} V_{\beta}$$

Then: (a) Q is invertible

(b) For all $\vec{v} \in V$, $[\vec{v}]_\beta = Q[\vec{v}]_{\beta'}$

Proposition: Let T be a linear operator on finite-dim V. Let β and β' be ordered bases of V. Suppose $Q = [I_V]_{\beta'}^{\beta}$.

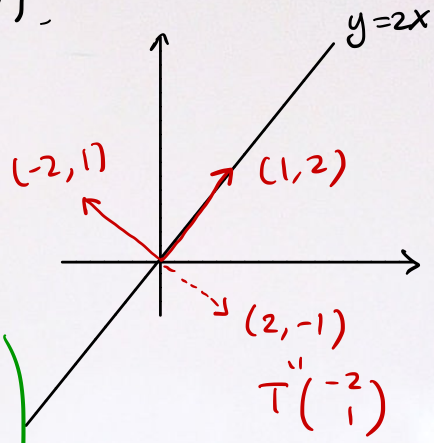
Then:
$$[T]_{\beta'} = Q^{-1}[T]_\beta Q$$

Remark: A linear $T: V \to V$ is called linear operator.

Example: Let $T:\mathbb{R}^2 \to \mathbb{R}^2$ be the reflection about the line $y=2x$.

Want to compute $[T]_\beta$, where $\beta = \left\{ \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix} \right\}$.

Consider $\beta' = \left\{ \begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}-2\\1\end{pmatrix} \right\}$ for $\mathbb{R}^2$



• $[T]_{\beta'} = \begin{pmatrix} | & | \\ \left[T\begin{pmatrix}1\\2\end{pmatrix}\right]_{\beta'} & \left[T\begin{pmatrix}-2\\1\end{pmatrix}\right]_{\beta'} \\ | & | \end{pmatrix}$

$$= \begin{pmatrix} | & | \\ \left[\begin{pmatrix}1\\2\end{pmatrix}\right]_{\beta'} & \left[\begin{pmatrix}2\\-1\end{pmatrix}\right]_{\beta'} \\ | & | \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

• $Q = [I_{\mathbb{R}^2}]_{\beta'}^{\beta} = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \Rightarrow Q^{-1} = \frac{1}{5}\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$

$$\therefore \quad [T]_{\beta'} = Q^{-1}[T]_\beta Q$$

$$\iff \quad [T]_\beta = Q \underset{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}{\underset{\shortparallel}{[T]_{\beta'}}} Q^{-1} = \frac{1}{5}\begin{pmatrix} -3 & 4 \\ 4 & 3 \end{pmatrix}.$$

<u>Def:</u> Given two matrices $A, B \in M_{n\times n}(F)$.
We say B is <u>similar</u> to A if $\exists\, Q \in M_{n\times n}$ s.t.

$$B = Q^{-1} A Q.$$

Dual Space

Let V be a vector space over F.

Definition: A linear functional on V is a linear map $f: V \to F$.

Remark: A linear functional belongs to $\mathcal{L}(V, F)$.

Definition: The dual space, denoted by V^*, is the space of all linear functional on V. That is, $V^* = \mathcal{L}(V, F)$.

Proposition: Suppose V is finite-dimensional. Let $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ be a basis of V. For each $i = 1, 2, \dots, n$, define a linear functional f_i by setting: $f_i(\vec{v}_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$

Then: $\{f_1, f_2, \dots, f_n\}$ is a basis of V^*, called the dual basis of $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$. $\therefore \dim(V) = \dim(V^*)$

Proof: • $\{f_1, f_2, \ldots, f_n\}$ is linearly independent.

Suppose: $a_1 f_1 + a_2 f_2 + \ldots + a_n f_n = 0$ ← zero functional

For each $\vec{v}_i$,

$$(a_1 f_1 + \ldots + a_n f_n)(\vec{v}_i) = 0 \Rightarrow a_1 f_1(\vec{v}_i) + \ldots + a_n f_n(\vec{v}_i) = 0$$

$$\Rightarrow a_i = 0.$$

$\therefore$ $\{f_1, f_2, \ldots, f_n\}$ is linearly independent.

• $\mathrm{Span}(\{f_1, f_2, \ldots, f_n\}) = V^*$.

Let $f \in V^*$. Suppose $f(\vec{v}_i) = b_i$.

Claim: $b_1 f_1 + b_2 f_2 + \ldots + b_n f_n = f$.

Check: For each $\vec{v}_i$,

$$(b_1 f_1 + \ldots + b_n f_n)(\vec{v}_i) = b_i \cancelto{1}{f_i(\vec{v}_i)} = f(\vec{v}_i) \Rightarrow b_1 f_1 + \ldots + b_n f_n = f.$$

<u>Example:</u> Let $\beta = \{1+x, 1-x, x^2\}$ be the ordered basis for $P_2(\mathbb{R})$

Let β^* be the dual basis of β.

$\beta^* = \{f_1, f_2, f_3\}$

Then:

$$1 = f_1(1+x) = f_1(1) + f_1(x)$$

$$0 = f_1(1-x) = f_1(1) - f_1(x)$$

$$0 = f_1(x^2)$$

Solving: we get $f_1(1) = \frac{1}{2}$, $f_1(x) = \frac{1}{2}$, $f_1(x^2) = 0$

Thus, $f_1(a+bx+cx^2) = a f_1(1) + b f_1(x) + c f_1(x^2)$

$$= \frac{1}{2}a + \frac{1}{2}b$$

f_2 and f_3 can be computed similarly.

Remark:

- $\dim(V) = \dim(V^*)$ $\therefore$ V is isomorphic to V^*

  fin-dim

- $V^{**} = (V^*)^* =$ dual of the dual space

Proposition: Suppose V is fin-dim. The map $\ell : V \to V^{**}$ defined by $\ell(\vec{v})(f) \overset{\text{def}}{=} f(\vec{v})$ is an isomorphism.

$\in V^*$

Proof: ℓ is linear (Exercise)

To prove that ℓ is an isomorphism, we can just show that $\vec{v}^*$ is $1-1$ (since $\dim(V) = \dim(V^{**})$)

Suppose $\ell(\vec{v}) = 0$ in V^{**}.

$\Rightarrow \ell(\vec{v})(f) = 0$ for all $f \in V^*$

Then: $f(\vec{v}) = 0$ for all $f \in V^*$

The only possibility is $\vec{v} = \vec{0}$.

(if $\vec{v} \neq \vec{0}$, then construct a basis $\{\vec{v}, \vec{v}_2, \ldots, \vec{v}_n\}$.)
Define $f \to f(\vec{v}) = 1$ and $f(\vec{v}_j) = 0$ for $j = 2, \ldots, n$)

$\therefore$ $\mathrm{Null}(\ell) = \{\vec{0}\}$. Thus, ℓ is 1-1 and onto.

(isomorphism)

Definition: Let $T: V \to W$ be linear. The dual map (or transpose) of T is the map $T^*: W^* \to V^*$ defined by:

$$T^*(g) = g(T) \quad \text{for all } g \in W^*.$$

Proposition: Suppose V is fin-dimensional. Let $\overset{\beta}{=}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ be a basis of V. Let $\beta^* = \{f_1, \dots, f_n\}$ be the dual basis of β. Let $T: V \to W$ and γ be the basis of W. Denote the dual basis of γ by γ^*. Then: (1) T^* is linear

(2)
$$[\underbrace{T}_{\text{Transpose of } T}{}^*]_{\gamma^*}^{\beta^*} = \underbrace{\left([T]_\beta^\gamma\right)^T}_{\text{Matrix transpose}}$$

$$\begin{tikzcd}
\underset{\beta}{V} \arrow[r, "T"] & \underset{\gamma}{W} \\
\underset{\beta^*}{V^*} & \underset{\gamma^*}{W^*} \arrow[l, "T^*"']
\end{tikzcd}$$

Proof: For any $g \in W^*$, $T^*(g) = \underbrace{g}_{\text{linear}} \circ \underbrace{T}_{\text{linear}}$ is linear.

$\therefore$ $T^*(g)$ is a linear functional on V. $\therefore$ $T^*(g) \in V^*$.

Thus: T^* maps W^* to V^*.

T^* is linear: $T^*(\alpha g_1 + g_2) = (\alpha g_1 + g_2) \circ T$

$$= \alpha g_1 \circ T + g_2 \circ T = \alpha T^*(g_1) + T^*(g_2)$$

Now, write $\beta = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$

$\gamma = \{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n\}$

$\beta^* = \{f_1, f_2, \ldots, f_n\}$

$\gamma^* = \{g_1, g_2, \ldots, g_n\}$

Let $A = [T]_\beta^\gamma = (a_{ij})$

To find the j^{th} col of $[T^*]_{\gamma^*}^{\beta^*}$, we write:

$T^*(g_j)$ as a lin. combination of $f_1, f_2, \dots, f_n$.

Now, $T^*(g_j) = g_j \circ T = \sum_{i=1} (g_j \circ T)(\vec{v}_i) f_i$

$\therefore$ the i^{th}-row, j^{th} col entry of $[T^*]_{\gamma^*}^{\beta^*}$ is given by:

$$g_j \circ T(\vec{v}_i) = g_j\left(\sum_{k=1}^{m} A_{ki} \vec{w}_k\right)$$

$$= \sum_{k=1}^{m} A_{ki}\, g_j(\vec{w}_k) = A_{ji}$$

$\therefore [T^*]_{\gamma^*}^{\beta^*} = A^T = [T]_\beta^\gamma$.