Math 3360: Mathematical Imaging

Midterm practice

Please give reasons in your solutions.

1. Recall that an image transformation $\mathcal{O} : M_{n\times n}(\mathbb{R}) \to M_{n\times n}(\mathbb{R})$ is said to be separable if there exist matrices $A \in M_{n\times n}(\mathbb{R})$ and $B \in M_{n\times m}(\mathbb{R})$ such that $\mathcal{O}(f) = AfB$ for any $f \in M_{n\times n}(\mathbb{R})$.

   Here $\mathcal{O} : M_{3\times 3}(\mathbb{R}) \to M_{3\times 3}(\mathbb{R})$ is an image transformation and the transformation matrix of its PSF is
$$H = \begin{pmatrix} 2 & 4 & 6 & 2 & 4 & 6 & 0 & 0 & 0 \\ 8 & 10 & 0 & 8 & 10 & 0 & 0 & 0 & 0 \\ 12 & 0 & 0 & 12 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 0 & 0 & 0 & 3 & 6 & 9 \\ 4 & 5 & 0 & 0 & 0 & 0 & 12 & 15 & 0 \\ 6 & 0 & 0 & 0 & 0 & 0 & 18 & 0 & 0 \\ 0 & 0 & 0 & 4 & 8 & 12 & 1 & 2 & 3 \\ 0 & 0 & 0 & 16 & 20 & 0 & 4 & 5 & 0 \\ 0 & 0 & 0 & 24 & 0 & 0 & 6 & 0 & 0 \end{pmatrix}.$$
   Please determine if $\mathcal{O}$ is separable. If yes, please find the corresponding matrices $A \in M_{3\times 3}(\mathbb{R})$ and $B \in M_{3\times 3}(\mathbb{R})$.

2. A matrix $H \in M_{n^2\times n^2}(\mathbb{R})$ is called block-circulant if it has the form
$$H = \begin{pmatrix} H_1 & H_n & \cdots & H_2 \\ H_2 & H_1 & \cdots & H_3 \\ \vdots & \vdots & \ddots & \vdots \\ H_n & H_{n-1} & \cdots & H_1 \end{pmatrix},$$
   where $H_i \in M_{n\times n}(\mathbb{R})$ for $i = 1, \cdots, n$. Given matrix $k, f \in M_{2\times 2}(\mathbb{R})$, let the image transformation $\mathcal{O}(f) = k * f$, please prove that the transformation matrix H of $\mathcal{O}$ is block-circulant.

3. Let $H = \begin{pmatrix} r & 2r & u & 2u \\ 3r & r & 3v & v \\ 3 & 6 & s & 2s \\ 9 & 3 & 3s & s \end{pmatrix}$ be the transformation matrix corresponding to an image transformation $\mathcal{O} : M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$, where r, s, u, v are all non-zero real numbers. Prove that $\mathcal{O}$ is separable and if and only if $u = v$. Please explain your answer with details.

4. Let $H = \begin{pmatrix} 2 & 4 & a & 6 \\ 4 & 2 & 6 & 1 \\ 1 & 6 & 2 & 4 \\ c & 1 & 4 & b \end{pmatrix}$ be the transformation matrix corresponding to an image transformation $\mathcal{O} : M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$, where a, b, c are all non-zero real numbers. Please determine a, b, c such that $\mathcal{O}$ is an image transformation defined by convolution.

5. Let $f = \begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 4 & 0 & 2 \end{pmatrix}$.

   (a) Compute an SVD of f.

   (b) Express f as a linear combination of its elementary images.

6. Let $f = \begin{pmatrix} 5 & 4 & 6 & 6 \\ 6 & 1 & 6 & 3 \\ 1 & 2 & 1 & 5 \\ 6 & 4 & 6 & 1 \end{pmatrix}$.

   (a) Compute the Haar transform f_{Haar} of f.

   (b) Suppose there is only enough capacity to store 10 pixel values of f_{Haar}. Choose 10 entries to keep such that the reconstucted image differs as little as possible in Frobenius norm with the original image, and compute the reconstructed image.

7. Let $H_n(t)$ be the n^{th} Haar function, where $n \in \mathbb{N} \cup \{0\}$.

   (a) Write down the definition of $H_n(t)$.

   (b) Write down the Haar transform matrix $\tilde{H}$ for 4×4 images.

   (c) Suppose $A = \begin{pmatrix} 0 & 1 & 1 & 2 \\ 2 & 3 & 3 & 4 \\ 2 & 3 & 3 & 4 \\ 4 & 5 & 5 & 6 \end{pmatrix}$. Compute the Haar transform A_{Haar} of A, and compute the reconstructed image $\tilde{A}$ after setting the largest entry of A_{Haar} to 0.

8. Suppose the definition of the DFT on $N \times N$ images is changed to

$$\hat{f}(m,n) = DFT(f)(m,n) = \frac{1}{N} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} f(k,l) e^{2\pi j \frac{mk+nl}{N}}.$$

   (a) Does there exist a matrix U such that $\hat{f} = UfU$ for an $N \times N$ image f? If yes, derive U and check if it is unitary.

   (b) Show that the inverse DFT (iDFT) is defined by

$$f(p,q) = iDFT(\hat{f})(p,q) = \frac{1}{N} \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} \hat{f}(m,n) e^{-2\pi j \frac{pm+qn}{N}}.$$

9. Let $f = \begin{pmatrix} 3 & 2 & 4 & 4 \\ 4 & -3 & 4 & 0 \\ -2 & -1 & -2 & 3 \\ 4 & 1 & 4 & -2 \end{pmatrix}$.

   (a) Compute the discrete Fourier transform $\hat{f}$ of f.

   (b) Compute the image reconstructed from $\hat{f}$ after removing frequencies in 3rd row and 3rd column.

10. Let $f, g \in M_{M \times N}(\mathbb{R})$ be periodically extended, please prove $\widehat{f * g} = MN \hat{f} \odot \hat{g}$, where $\hat{f} \odot \hat{g}(m,n) = \hat{f}(m,n)\hat{g}(m,n)$.

11. Let $f, g \in M_{M \times N}(\mathbb{R})$ be periodically extended, please prove $\widehat{f \odot g} = \hat{f} * \hat{g}$, where $f \odot g(k,l) = f(k,l)g(k,l)$.

12. Let $f \in M_{N \times N}(\mathbb{R})$ be periodically extended, and let $\tilde{f}(k,l) = f(l,-k)$, please prove $\hat{\tilde{f}} = \tilde{\hat{f}}$.

13. Let $f \in M_{M \times N}(\mathbb{R})$ be periodically extended, and let $\tilde{f}(k,l) = f(k-k_0, l-l_0)$ for some $k_0, l_0 \in \mathbb{Z}$, please prove $\hat{\tilde{f}} = e^{-2\pi j(\frac{k_0 m}{M} + \frac{l_0 n}{N})} \hat{f}$.

14. Let $f \in M_{M \times N}(\mathbb{R})$ be periodically extended, and let $\tilde{\hat{f}}(m,n) = \hat{f}(m-m_0, n-n_0)$ for some $m_0, n_0 \in \mathbb{Z}$, please prove $\tilde{\hat{f}} = DFT(e^{2\pi j(\frac{km_0}{M} + \frac{ln_0}{N})} f)$.

15. Please prove that the rank k approximation is the optimal approximation for rank k matrix in sense of Frobenius norm. That is, given a rank r matrix $A \in M_{n\times m}(\mathbb{R})$, for any rank k matrix $B \in M_{n\times m}(\mathbb{R})$, we have

$$\|A - B\|_F \geq \|A - A_k\|_F,$$

where $A_k = \sum_{i=1}^{k} \sigma_i \vec{u_i}\vec{v_i}^T$ is the rank k approximation of $A = \sum_{i=1}^{r} \sigma_i \vec{u_i}\vec{v_i}^T$ and $k = 1, 2, \cdots, r$.