MATH3360 Mathematical Imaging
Midterm Examination

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You have to answer all five questions. The total score is **100**.
Please show your steps unless otherwise stated.

1. Let $\mathcal{O} : M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ be a linear image transformation.

   (a) Suppose $\mathcal{O}$ is separable and $\mathcal{O}(f) = AfB$ for any $f \in M_{2\times 2}(\mathbb{R})$, where $A, B \in M_{2\times 2}(\mathbb{R})$. Prove that the transformation matrix of $\mathcal{O}$ is $H = B^T \otimes A$.

   (b) Let
   $$H = \begin{pmatrix} a & 4 & 4 & 2 \\ 2 & a-2 & b & 3 \\ c & 2 & 12 & 9-d \\ 1 & 3 & d & 9 \end{pmatrix}$$
   be the transformation matrix corresponding to $\mathcal{O}$. Determine suitable $a, b, c, d \in \mathbb{R}$ such that $\mathcal{O}$ is separable and find matrices A and B.

   (c) Suppose $\mathcal{O}$ is defined by convolution and $\mathcal{O}(f) = h * f$ for any $f \in M_{2\times 2}(\mathbb{R})$, where $h \in M_{2\times 2}(\mathbb{R})$. Prove that the transformation matrix H of $\mathcal{O}$ is block circulant, i.e. $H = \begin{pmatrix} H_1 & H_2 \\ H_2 & H_1 \end{pmatrix}$, where $H_1, H_2 \in M_{2\times 2}(\mathbb{R})$.

   Solution:

   (a) Let $A = (a_{ij})_{1\le i,j\le 2}$, $B = (b_{ij})_{1\le i,j\le 2}$ and $g = \mathcal{O}(f) \in M_{2\times 2}(\mathbb{R})$, then we have
   $$g_{\alpha,\beta} = \sum_{x=1}^{2} a_{\alpha x}(\sum_{y=1}^{2} f(x,y)b_{y\beta}) = \sum_{x=1}^{2}\sum_{y=1}^{2} a_{\alpha x} b_{y\beta} f(x,y),$$
   Which means $h^{\alpha,\beta}(x,y) = a_{\alpha x} b_{y\beta}$. Hence the transformation matrix
   $$H = \begin{pmatrix} a_{11}b_{11} & a_{12}b_{11} & a_{11}b_{21} & a_{12}b_{21} \\ a_{21}b_{11} & a_{22}b_{11} & a_{21}b_{21} & a_{22}b_{21} \\ a_{11}b_{12} & a_{12}b_{12} & a_{11}b_{22} & a_{12}b_{22} \\ a_{21}b_{12} & a_{22}b_{12} & a_{21}b_{22} & a_{22}b_{22} \end{pmatrix} = \begin{pmatrix} b_{11}A & b_{21}A \\ b_{12}A & b_{22}A \end{pmatrix} = B^T \otimes A.$$

   (b) We have
   $$a/4 = (a-2)/3 = 2/b = 4/2,$$
   $$c/12 = 2/(9-d) = 1/d = 3/9.$$
   Therefore $a = 8, b = 1, c = 4, d = 3$ and
   $$H = \begin{pmatrix} 8 & 4 & 4 & 2 \\ 2 & 6 & 1 & 3 \\ 4 & 2 & 12 & 6 \\ 1 & 3 & 3 & 9 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} \otimes \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix} = B^T \otimes A.$$
   i.e. $A = \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$.

   (c) Let $k = \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix}$ and $g = \mathcal{O}(f) \in M_{2\times 2}(\mathbb{R})$. Then we have
   $$g_{\alpha,\beta} = \sum_{x=1}^{2}\sum_{y=1}^{2} k_{\alpha-x,\beta-y} f(x,y),$$

which means $h^{\alpha,\beta}(x,y) = k_{\alpha-x,\beta-y}$. Hence the transformation matrix

$$H = \begin{pmatrix} k_{22} & k_{12} & k_{21} & k_{11} \\ k_{12} & k_{22} & k_{11} & k_{21} \\ k_{21} & k_{11} & k_{22} & k_{12} \\ k_{11} & k_{21} & k_{12} & k_{22} \end{pmatrix} = \begin{pmatrix} H_1 & H_2 \\ H_2 & H_1 \end{pmatrix}$$

where $H_1 = \begin{pmatrix} k_{22} & k_{12} \\ k_{12} & k_{22} \end{pmatrix}$ and $H_2 = \begin{pmatrix} k_{21} & k_{11} \\ k_{11} & k_{21} \end{pmatrix}$. Hence the transformation matrix H of $\mathcal{O}$ is block-circulant.

2. Let

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

(a) Compute SVD of A. Express A as a linear combination of its elementary images.

(b) Find a rank 2 approximation A_2 such that $\|A_2 - A\|_F = 2$. Please prove your answer with details.

(c) Using (a), determine the SVD of a distorted image $\widetilde{A} = \begin{pmatrix} 1+\epsilon & 2+\tau & 0 \\ 2+\tau & 1+\epsilon & 0 \\ 0 & 0 & 2+\tau \end{pmatrix}$, where ϵ and τ are small positive real numbers. Please explain your answer with details.

Solution:

(a) We first compute the characteristic polynomial of $AA^T = \begin{pmatrix} 5 & 4 & 0 \\ 4 & 5 & 0 \\ 0 & 0 & 4 \end{pmatrix}$. The characteristic polynomial of A^TA is given by

$$\det(A^TA - \lambda E) = \begin{vmatrix} 5-\lambda & 4 & 0 \\ 4 & 5-\lambda & 0 \\ 0 & 0 & 4-\lambda \end{vmatrix}$$
$$= (4-\lambda)((5-\lambda)^2 - 16) = (9-\lambda)(4-\lambda)(1-\lambda).$$

So the eigenvalues of A^TA is $\lambda_1 = 9$, $\lambda_2 = 4$ and $\lambda_3 = 1$. The corresponding eigenvectors are

$$\vec{v}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \text{ and } \vec{v}_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$

Then we have

$$\vec{u}_1 = \frac{A^T\vec{v}_1}{\sqrt{\lambda_1}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \vec{u}_2 = \frac{A^T\vec{v}_2}{\sqrt{\lambda_2}} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \text{ and } \vec{u}_3 = \frac{A^T\vec{v}_3}{\sqrt{\lambda_3}} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}.$$

Therefore, $A = U\Sigma V^T$, where

$$U = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix}, \Sigma = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } V = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix}.$$

The eigenimages are given by

$$\vec{u}_1\vec{v}_1^T = \frac{1}{2}\begin{pmatrix}1\\1\\0\end{pmatrix}\begin{pmatrix}1 & 1 & 0\end{pmatrix} = \begin{pmatrix}\frac{1}{2} & \frac{1}{2} & 0\\ \frac{1}{2} & \frac{1}{2} & 0\\ 0 & 0 & 0\end{pmatrix} \text{ and}$$

$$\vec{u}_2\vec{v}_2^T = \begin{pmatrix}0\\0\\1\end{pmatrix}\begin{pmatrix}0 & 0 & 1\end{pmatrix} = \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}.$$

$$\vec{u}_3\vec{v}_3^T = \frac{1}{2}\begin{pmatrix}-1\\1\\0\end{pmatrix}\begin{pmatrix}1 & -1 & 0\end{pmatrix} = \begin{pmatrix}-\frac{1}{2} & \frac{1}{2} & 0\\ \frac{1}{2} & -\frac{1}{2} & 0\\ 0 & 0 & 0\end{pmatrix}.$$

Hence $A = 3\begin{pmatrix}\frac{1}{2} & \frac{1}{2} & 0\\ \frac{1}{2} & \frac{1}{2} & 0\\ 0 & 0 & 0\end{pmatrix} + 2\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{pmatrix} + \begin{pmatrix}-\frac{1}{2} & \frac{1}{2} & 0\\ \frac{1}{2} & -\frac{1}{2} & 0\\ 0 & 0 & 0\end{pmatrix}.$

(b) Since $A = U\Sigma V^T$, let $A_2 = U\Sigma_2 V^T$, where

$$\Sigma_2 = \begin{pmatrix}3 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}.$$

It's clear that $\text{rank}(A_2) = 2$ and

$$\|A_2 - A\|_F = \|U(\Sigma_2 - \Sigma)V^T\|_F = \|\Sigma_2 - \Sigma\|_F = 2.$$

(c) Using (a), we can also express $\widetilde{A}$ by elementary images that

$$\widetilde{A} = A + \begin{pmatrix}\epsilon & \tau & 0\\ \tau & \epsilon & 0\\ 0 & 0 & \tau\end{pmatrix} = A + (\epsilon+\tau)\begin{pmatrix}\frac{1}{2} & \frac{1}{2} & 0\\ \frac{1}{2} & \frac{1}{2} & 0\\ 0 & 0 & 0\end{pmatrix} + \tau\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{pmatrix} + (\tau-\epsilon)\begin{pmatrix}-\frac{1}{2} & \frac{1}{2} & 0\\ \frac{1}{2} & -\frac{1}{2} & 0\\ 0 & 0 & 0\end{pmatrix}$$

$$= (3+\epsilon+\tau)\begin{pmatrix}\frac{1}{2} & \frac{1}{2} & 0\\ \frac{1}{2} & \frac{1}{2} & 0\\ 0 & 0 & 0\end{pmatrix} + (2+\tau)\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{pmatrix} + (1+\tau-\epsilon)\begin{pmatrix}-\frac{1}{2} & \frac{1}{2} & 0\\ \frac{1}{2} & -\frac{1}{2} & 0\\ 0 & 0 & 0\end{pmatrix}.$$

Therefore, the SVD is $\widetilde{A} = U\widetilde{\Sigma}V^T$, where $\widetilde{\Sigma} = \begin{pmatrix}3+\epsilon+\tau & 0 & 0\\ 0 & 2+\tau & 0\\ 0 & 0 & 1+\tau-\epsilon\end{pmatrix}.$

3. Recall that the 0-th Haar function is

$$H_0(t) = \begin{cases}1 & \text{if } 0 \le t < 1\\ 0 & \text{otherwise,}\end{cases}$$

and the other Haar functions are defined by

$$H_{2^p+n}(t) = \begin{cases}\sqrt{2}^p & \text{if } \frac{n}{2^p} \le t < \frac{n+0.5}{2^p}\\ -\sqrt{2}^p & \text{if } \frac{n+0.5}{2^p} \le t < \frac{n+1}{2^p}\\ 0 & \text{otherwise}\end{cases}$$

for $p = 0, 1, 2, \cdots$ and $n = 0, 1, 2, \cdots, 2^p - 1$.

(a) Give the definition of Haar transformation for 4×4 images.

(b) Suppose

$$A = \begin{pmatrix}5 & 0 & 0 & 3\\ 2 & 1 & 1 & 4\\ 2 & 1 & 1 & 4\\ 4 & 0 & 0 & 6\end{pmatrix},$$

please compute the Haar transform A_{Haar} of A.

(c) Suppose A_{Haar} is corrupted by noise, such that

$$\tilde{A}_{\text{Haar}} = A_{\text{Haar}} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \epsilon \end{pmatrix}.$$

Let $\widetilde{A}$ be the reconstructed image from $\tilde{A}_{\text{Haar}}$. Discuss which pixels of $\widetilde{A}$ have different intensity values from A. Find the error $||\widetilde{A} - A||_F$.

Solution:

(a) $\tilde{H} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{pmatrix}$.

The Haar Transform of $I \in M_{4\times 4}(\mathbb{R})$ is given by $\tilde{H} I \tilde{H}^T$.

(b)

$$\begin{aligned} A_{\text{Haar}} &= \tilde{H} A \tilde{H}^T \\ &= \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{pmatrix}\begin{pmatrix} 5 & 0 & 0 & 3 \\ 2 & 1 & 1 & 4 \\ 2 & 1 & 1 & 4 \\ 4 & 0 & 0 & 6 \end{pmatrix}\begin{pmatrix} 1 & 1 & \sqrt{2} & 0 \\ 1 & 1 & -\sqrt{2} & 0 \\ 1 & -1 & 0 & \sqrt{2} \\ 1 & -1 & 0 & -\sqrt{2} \end{pmatrix} \\ &= \frac{1}{4}\begin{pmatrix} 13 & 2 & 2 & 17 \\ 1 & 0 & 0 & -3 \\ 3\sqrt{2} & -\sqrt{2} & -\sqrt{2} & -\sqrt{2} \\ -2\sqrt{2} & \sqrt{2} & \sqrt{2} & -2\sqrt{2} \end{pmatrix}\begin{pmatrix} 1 & 1 & \sqrt{2} & 0 \\ 1 & 1 & -\sqrt{2} & 0 \\ 1 & -1 & 0 & \sqrt{2} \\ 1 & -1 & 0 & -\sqrt{2} \end{pmatrix} \\ &= \frac{1}{4}\begin{pmatrix} 34 & -4 & 11\sqrt{2} & -15\sqrt{2} \\ -2 & 4 & \sqrt{2} & 3\sqrt{2} \\ 0 & 4\sqrt{2} & 8 & 0 \\ -2\sqrt{2} & 0 & -6 & 6 \end{pmatrix}. \end{aligned}$$

(c) We have

$$\begin{aligned} \widetilde{A} = \widetilde{H}^T \tilde{A}_{\text{Haar}} \widetilde{H} &= \widetilde{H}^T A_{\text{Haar}} \widetilde{H} + \widetilde{H}^T \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \epsilon \end{pmatrix} \widetilde{H} \\ &= A + \frac{1}{4}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2\epsilon & -2\epsilon \\ 0 & 0 & -2\epsilon & 2\epsilon \end{pmatrix} = \begin{pmatrix} 5 & 0 & 0 & 3 \\ 2 & 1 & 1 & 4 \\ 2 & 1 & 1+\frac{1}{2}\epsilon & 4-\frac{1}{2}\epsilon \\ 4 & 0 & -\frac{1}{2}\epsilon & 6+\frac{1}{2}\epsilon \end{pmatrix}. \end{aligned}$$

Therefore, it's clear that only when $i, j \in \{3, 4\}$, the ij entries of $\widetilde{A}$ changes and

$$\|\widetilde{A} - A\|_F = \left\| \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2}\epsilon & -\frac{1}{2}\epsilon \\ 0 & 0 & -\frac{1}{2}\epsilon & \frac{1}{2}\epsilon \end{pmatrix} \right\|_F = |\epsilon|.$$

4. Let $I \in M_{N\times N}(\mathbb{R})$ be a $N \times N$ image, whose indices are taken from 0 to $N-1$. Assuming that I is periodically extended. Let k_0, l_0 be two integers between 0 and

$N-1$. Recall that the discrete Fourier transform (DFT) of an $N \times N$ image I is defined as

$$DFT(I)(m,n) = \frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} I(k,l) e^{-2\pi\sqrt{-1}(\frac{km+ln}{N})}$$

for all $0 \leq m, n \leq N-1$.

(a) Consider another image I_1 defined by: $I_1(k,l) = I(k-k_0, l-l_0)$ for all $0 \leq k, l \leq N-1$. Write $DFT(I_1)$ in terms of $DFT(I)$.

(b) Consider another image I_2 defined by: $I_2(k,l) = I(-l+l_0, -k+k_0)$ for all $0 \leq k, l \leq N-1$. Write $DFT(I_2)$ in terms of $DFT(I)$.

Please show all your steps clearly (including how the changes of variables are applied, indices are shifted and so on). Missing details will lead to mark deduction.

Solution: Write $j = \sqrt{-1}$.

(a)

$$\begin{aligned}
DFT(I_1)(m,n) &= \frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} I_1(k,l) e^{-\frac{2\pi j}{N}(km+ln)} \\
&= \frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} I(k-k_0, l-l_0) e^{-\frac{2\pi j}{N}(km+ln)} \\
\text{(letting } l' = l - l_0) \quad &= \frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{l'=-l_0}^{N-1-l_0} I(k-k_0, l') e^{-\frac{2\pi j}{N}(km+(l'+l_0)n)} \\
&= \frac{e^{-\frac{2\pi j}{N}(l_0 n)}}{N^2} \sum_{k=0}^{N-1} \sum_{l'=-l_0}^{N-1-l_0} I(k-k_0, l') e^{-\frac{2\pi j}{N}(km+l'n)} \\
&= \frac{e^{-\frac{2\pi j}{N}(l_0 n)}}{N^2} \sum_{k=0}^{N-1} \left[\sum_{l'=0}^{N-1-l_0} I(k-k_0, l') e^{-\frac{2\pi j}{N}(km+l'n)} \right. \\
&\quad \left. + \sum_{l'=-l_0}^{-1} I(k-k_0, l') e^{-\frac{2\pi j}{N}(km+l'n)} \right] \\
\text{(letting } l'' = l' + N) \quad &= \frac{e^{-\frac{2\pi j}{N}(l_0 n)}}{N^2} \sum_{k=0}^{N-1} \left[\sum_{l'=0}^{N-1-l_0} I(k-k_0, l') e^{-\frac{2\pi j}{N}(km+l'n)} \right. \\
&\quad \left. + \sum_{l''=N-l_0}^{N-1} I(k-k_0, l''-N) e^{-\frac{2\pi j}{N}(km+(l''-N)n)} \right] \\
\text{(by periodicity and } e^{2\pi j n} = 1) \quad &= \frac{e^{-\frac{2\pi j}{N}(l_0 n)}}{N^2} \sum_{k=0}^{N-1} \left[\sum_{l'=0}^{N-1-l_0} I(k-k_0, l') e^{-\frac{2\pi j}{N}(km+l'n)} \right. \\
&\quad \left. + \sum_{l''=N-l_0}^{N-1} I(k-k_0, l'') e^{-\frac{2\pi j}{N}(km+l''n)} \right] \\
\text{(rewrite } l' \text{ and } l'' \text{ as } l) \quad &= \frac{e^{-\frac{2\pi j}{N}(l_0 n)}}{N^2} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} I(k-k_0, l) e^{-\frac{2\pi j}{N}(km+ln)}
\end{aligned}$$

(cont.)

$$
\begin{aligned}
\text{(letting } k' = k - k_0\text{)} \quad &= \frac{1}{N^2} \sum_{k'=-k_0}^{N-1-k_0} \sum_{l=0}^{N-1} I(k', l) e^{-\frac{2\pi j}{N}((k'+k_0)m+ln)} \\
&= \frac{e^{-\frac{2\pi j}{N}(k_0 m + l_0 n)}}{N^2} \sum_{k'=-k_0}^{N-1-k_0} \sum_{l=0}^{N-1} I(k', l) e^{-\frac{2\pi j}{N}(k'm+ln)} \\
&= \frac{e^{-\frac{2\pi j}{N}(k_0 m + l_0 n)}}{N^2} \sum_{l=0}^{N-1} \Bigg[\sum_{k'=0}^{N-1-k_0} I(k', l) e^{-\frac{2\pi j}{N}(k'm+ln)} \\
&\quad + \sum_{k'=-k_0}^{-1} I(k', l) e^{-\frac{2\pi j}{N}(k'm+ln)} \Bigg] \\
\text{(letting } k'' = k' + N\text{)} \quad &= \frac{e^{-\frac{2\pi j}{N}(k_0 m + l_0 n)}}{N^2} \sum_{l=0}^{N-1} \Bigg[\sum_{k'=0}^{N-1-k_0} I(k', l) e^{-\frac{2\pi j}{N}(k'm+ln)} \\
&\quad + \sum_{k''=N-k_0}^{N-1} I(k'' - N, l) e^{-\frac{2\pi j}{N}((k''-N)m+ln)} \Bigg] \\
\text{(by periodicity and } e^{2\pi j m} = 1\text{)} \quad &= \frac{e^{-\frac{2\pi j}{N}(k_0 m + l_0 n)}}{N^2} \sum_{l=0}^{N-1} \Bigg[\sum_{k'=0}^{N-1-k_0} I(k', l) e^{-\frac{2\pi j}{N}(k'm+ln)} \\
&\quad + \sum_{k''=N-k_0}^{N-1} I(k'', l) e^{-\frac{2\pi j}{N}(k''m+ln)} \Bigg] \\
\text{(rewrite } k' \text{ and } k'' \text{ as } k\text{)} \quad &= \frac{e^{-\frac{2\pi j}{N}(k_0 m + l_0 n)}}{N^2} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} I(k, l) e^{-\frac{2\pi j}{N}(km+ln)} \\
&= DFT(I)(m, n) e^{-\frac{2\pi j}{N}(k_0 m + l_0 n)}
\end{aligned}
$$

(b)

$$
\begin{aligned}
DFT(I_2)(m,n) &= \frac{1}{N^2}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1} I_2(k,l)e^{-\frac{2\pi j}{N}(km+ln)} \\
&= \frac{1}{N^2}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1} I(-l+l_0,-k+k_0)e^{-\frac{2\pi j}{N}(km+ln)} \\
\text{(letting } l' = -l+l_0) \quad &= \frac{1}{N^2}\sum_{k=0}^{N-1}\sum_{l'=l_0}^{l_0-N+1} I(l',-k+k_0)e^{-\frac{2\pi j}{N}(km+(-l'+l_0)n)} \\
&= \frac{e^{-\frac{2\pi j}{N}(l_0 n)}}{N^2}\frac{1}{N^2}\sum_{k=0}^{N-1}\sum_{l'=l_0-N+1}^{l_0} I(l',-k+k_0)e^{-\frac{2\pi j}{N}(km+(-l')n)} \\
&= \frac{e^{-\frac{2\pi j}{N}(l_0 n)}}{N^2}\frac{1}{N^2}\sum_{k=0}^{N-1}\left[\sum_{l'=0}^{l_0} I(l',-k+k_0)e^{-\frac{2\pi j}{N}(km+(-l')n)}\right. \\
&\quad \left. + \sum_{l'=l_0-N+1}^{-1} I(l',-k+k_0)e^{-\frac{2\pi j}{N}(km+(-l')n)}\right] \\
\text{(letting } l'' = l'+N) \quad &= \frac{e^{-\frac{2\pi j}{N}(l_0 n)}}{N^2}\frac{1}{N^2}\sum_{k=0}^{N-1}\left[\sum_{l'=0}^{l_0} I(l',-k+k_0)e^{-\frac{2\pi j}{N}(km+(-l')n)}\right. \\
&\quad \left. + \sum_{l''=l_0+1}^{N-1} I(l''-N,-k+k_0)e^{-\frac{2\pi j}{N}(km+(-(l''-N))n)}\right] \\
\text{(by periodicity and } e^{-2\pi jn}=1) \quad &= \frac{e^{-\frac{2\pi j}{N}(l_0 n)}}{N^2}\frac{1}{N^2}\sum_{k=0}^{N-1}\left[\sum_{l'=0}^{l_0} I(l',-k+k_0)e^{-\frac{2\pi j}{N}(km+(-l')n)}\right. \\
&\quad \left. + \sum_{l''=l_0+1}^{N-1} I(l'',-k+k_0)e^{-\frac{2\pi j}{N}(km+(-l'')n)}\right] \\
\text{(write } l' \text{ and } l'' \text{ as } l) \quad &= \frac{e^{-\frac{2\pi j}{N}(l_0 n)}}{N^2}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1} I(l,-k+k_0)e^{-\frac{2\pi j}{N}(km+(-l)n)}
\end{aligned}
$$

(cont.)

$$
\begin{aligned}
\text{(letting } k' = -k + k_0) \quad &= \frac{e^{-\frac{2\pi j}{N}(l_0 n)}}{N^2} \sum_{k'=k_0}^{k_0-N+1} \sum_{l=0}^{N-1} I(l,k') e^{-\frac{2\pi j}{N}((-k'+k_0)m+(-l)n)} \\
&= \frac{e^{-\frac{2\pi j}{N}(l_0 n+k_0 m)}}{N^2} \sum_{k'=k_0-N+1}^{k_0} \sum_{l=0}^{N-1} I(l,k') e^{-\frac{2\pi j}{N}((-k')m+(-l)n)} \\
&= \frac{e^{-\frac{2\pi j}{N}(l_0 n+k_0 m)}}{N^2} \sum_{l=0}^{N-1} \left[\sum_{k'=0}^{k_0} I(l,k') e^{-\frac{2\pi j}{N}((-k')m+(-l)n)} \right. \\
&\quad \left. + \sum_{k'=k_0-N+1}^{-1} I(l,k') e^{-\frac{2\pi j}{N}((-k')m+(-l)n)} \right] \\
\text{(letting } k'' = k' + N) \quad &= \frac{e^{-\frac{2\pi j}{N}(l_0 n+k_0 m)}}{N^2} \sum_{l=0}^{N-1} \left[\sum_{k'=0}^{k_0} I(l,k') e^{-\frac{2\pi j}{N}((-k')m+(-l)n)} \right. \\
&\quad \left. + \sum_{k''=k_0-1}^{N-1} I(l,k''-N) e^{-\frac{2\pi j}{N}((-k''-N)m+(-l)n)} \right] \\
\text{(by periodicity and } e^{-2\pi j m} = 1) \quad &= \frac{e^{-\frac{2\pi j}{N}(l_0 n+k_0 m)}}{N^2} \sum_{l=0}^{N-1} \left[\sum_{k'=0}^{k_0} I(l,k') e^{-\frac{2\pi j}{N}((-k')m+(-l)n)} \right. \\
&\quad \left. + \sum_{k''=k_0-1}^{N-1} I(l,k'') e^{-\frac{2\pi j}{N}((-k'')m+(-l)n)} \right] \\
\text{(write } k' \text{ and } k'' \text{ as } k) \quad &= \frac{e^{-\frac{2\pi j}{N}(l_0 n+k_0 m)}}{N^2} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} I(l,k) e^{-\frac{2\pi j}{N}((-k)m+(-l)n)} \\
&= \frac{e^{-\frac{2\pi j}{N}(l_0 n+k_0 m)}}{N^2} \sum_{l=0}^{N-1} \sum_{k=0}^{N-1} I(l,k) e^{-\frac{2\pi j}{N}(l(-n)+k(-m))} \\
&= DFT(I)(-n,-m) e^{-\frac{2\pi j}{N}(l_0 n+k_0 m)}
\end{aligned}
$$

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