Image denoising in the spatial domain

Definition: Linear filter = modify pixel value by a linear combination of pixel values of local neighbourhood.

Example 1: Let f be an $N \times N$ image. Extend the image periodically. Modify f to $\tilde{f}$ by:

$$\tilde{f}(x,y) = f(x,y) + 3f(x-1,y) + 2f(x+1,y).$$

This is a linear filter.

Example 2: Define

$$\tilde{f}(x,y) = \frac{1}{4}\left(f(x+1,y) + f(x-1,y) + f(x,y+1) + f(x,y-1)\right)$$

This is also a linear filter.

<u>Recall</u>: The discrete convolution is defined as:

$$I * H(u,v) = \sum_{m=-M}^{M} \sum_{n=-N}^{N} I(u-m, v-n) H(m,n)$$

(Linear combination of pixel values around (u,v))

Therefore, **Linear filter is equivalent to a discrete convolution**.

Geometric illustration

$$\begin{tikzcd}
I \arrow[d, "*\,H"] \arrow[dr, phantom, "\cong"] & \hat{I} \arrow[d, "\times\,\hat{H}"] \\
\tilde{I} \arrow[r, shift left, "DFT"] & \hat{\tilde{I}} \arrow[l, shift left, "iDFT"]
\end{tikzcd}$$

(spatial) (frequency)

Example 3: In Example 1, if f is defined on $[-M, M] \times [-N, N]$, then:

$$\tilde{f} = f * H$$

where

$$H = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 0 \end{pmatrix}$$

In Example 2, $\tilde{f} = f * H$ where

$$H = \frac{1}{4}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

H is called the <u>filter</u>

Commonly used filter (linear)

- **Mean filter:**

$$H = \frac{1}{9}\begin{array}{c} \begin{array}{ccc} -1 & 0 & 1 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \end{array} \begin{array}{c} \\ -1 \\ 0 \\ 1 \end{array}$$

(Here, we only write down the entries of the matrix for indices $-1 \leq k, l \leq 1$ for simplicity. All other matrix entries are equal to 0.)

This is called the *mean filtering with window size* 3×3.

- **Gaussian filter:** The entries of H are given by the Gaussian function $g(r) = exp\left(-\frac{r^2}{2\sigma^2}\right)$, where $r = \sqrt{x^2 + y^2}$.

Properties of linear filtering

- **Associativity:** $A * (B * C) = (A * B) * C$
- **Commutativity:** $I * H = H * I$
- **Linearity:**

$$(s \cdot I) * H = I * (s \cdot H) = s \cdot (I * H)$$

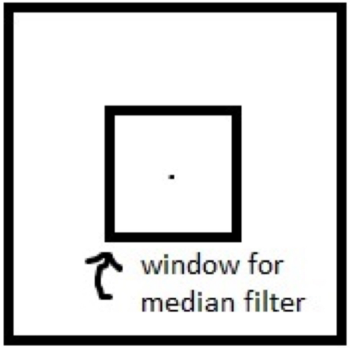
$$(I_1 + I_2) * H = (I_1 * H) + (I_2 * H)$$

Remark: Convolution of Gaussian with a Gaussian is also a Gaussian

$\therefore$ Successive Gaussian filter = Gaussian filter with larger σ.

Non-linear spatial filter

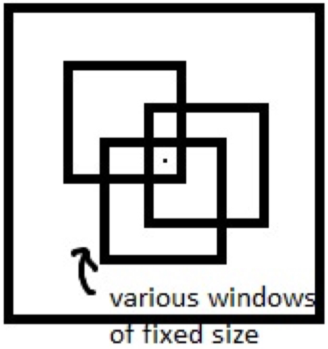
- **Median filter**



Take a window with center at pixel (x_0, y_0). Update the pixel value at (x_0, y_0) from $I(x_0, y_0)$ to $\tilde{I}(x_0, y_0) = \text{median}(I \text{ within the window})$

Example 4: If pixel values within a window is $0, 0, 1, 2, 3, 7, 8, 9, 9$, then the pixel value is updated as 3 (median).

- **Edge-preserving filter**



- **Step 1:** Consider all windows with certain size around pixel (x_0, y_0) (not necessarily be centered at (x_0, y_0));
- **Step 2:** Select a window with minimal variance;
- **Step 3:** Do a linear filter (mean filter, Gaussian filter and so on).