Math 3360: Mathematical Imaging

Assignment 4

Due: November 10 before 1159PM

Please give reasons in your solutions.

1. Consider a $2N \times 2N$ image $I = (I(m,n))_{-N \leq m,n \leq N-1}$. The Butterworth high-pass filter H of squared radius D_0 and order n is applied on $DFT(I) = (\hat{I}(u,v))_{-N \leq u,v \leq N-1}$ to give $G(u,v)$. Suppose $\hat{I}(1,1) \neq 0$ and $\hat{I}(2,2) \neq 0$, and
$$G(1,1) = \frac{1}{37}\hat{I}(1,1) \text{ and } G(2,2) = \frac{1}{10}\hat{I}(2,2).$$
Find D_0 and n.

2. (a) Consider a $(2M+1) \times (2N+1)$ image $I = (I(m,n))_{0 \leq m \leq 2M,\, 0 \leq n \leq 2N}$, where $M, N > 200$. The Gaussian high-pass filter with standard deviation σ is applied to $DFT(I) = (\hat{I}(u,v))_{0 \leq m \leq 2M,\, 0 \leq n \leq 2N}$. Suppose $H(2,2) = \frac{1}{MN}$. Find σ^2.

   (b) Consider a Gaussian low-pass filter
$$H(u,v) = exp\left(-\frac{u^2+v^2}{2\sigma^2}\right).$$
Suppose $H(4,2) = \frac{1}{\sqrt{e}}H(1,-3)$. Find σ^2.

3. Suppose $g \in M_{N \times N}(\mathbb{R})$ is a blurred image capturing a static scene. Assume that g is given by:
$$g(i,j) = \frac{1}{\lambda}\sum_{k=0}^{\lambda-1} f(i-k,j) \text{ for } 0 \leq i,j \leq N-1,$$
where $\lambda \in \mathbb{N} \cap [1,N]$ and f is the underlying image (periodically extended). Show that $DFT(g)(u,v) = H(u,v)DFT(f)(u,v)$ for all $0 \leq u,v \leq N-1$, where $H(u,v)$ is the degradation function in the frequency domain given by:
$$H(u,v) = \begin{cases} \frac{1}{\lambda}\frac{\sin\frac{\lambda\pi u}{N}}{\sin\frac{\pi u}{N}} e^{-\pi j \frac{(\lambda-1)u}{N}} & \text{if } u \neq 0, \\ 1 & \text{if } u = 0. \end{cases}$$

4. Consider a 4×4 periodically extended image $I = (I(k,l))_{0 \leq k,l \leq 3}$ given by:
$$I = \begin{pmatrix} a & a-2c & 0 & 0 \\ b-2c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b \end{pmatrix},$$
where $a, b, c \geq 0$.
Consider the modified direct filter T_1 with squared radius a and order b, which is defined by
$$T_1(u,v) = \frac{B(u,v)}{H_1(u,v) + \epsilon \cdot sgn(H_1(u,v))}$$
where $B(u,v) = \frac{1}{1+(\frac{u^2+v^2}{a})^b}$, $\epsilon = 1$ and $H_1(u,v) = DFT(h_1)(u,v)$ with h_1 being a bluring convolution kernel $\begin{pmatrix} 1/2 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

Consider the constrained least square filter T_2 with parameter c, which is defined by

$$T_2(u,v) = \frac{1}{N^2}\frac{\overline{H_2(u,v)}}{|H_2(u,v)|^2 + c|P(u,v)|^2}$$

where $H_2 = DFT(h_2)$ with h_2 being the convolution kernel $\begin{pmatrix} 1/3 & 1/3 & 1/3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. and $P = DFT(\tilde{h})$ with $\tilde{h}$ being the convolution kernel $\begin{pmatrix} -4 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$.

Let $I_2(u,v) = T_2(u,v)DFT(I)(u,v)$

Suppose $DFT(I)(0,0) = \frac{1}{2}$; $T_1(0,2) = \frac{8}{17}$; $DFT(I)(1,1) \neq 0$ and $I_2(1,1) = \frac{3j}{73}DFT(I)(1,1)$.
Find a, b, c. (Here, $j = \sqrt{-1}$.)

5. The constrained least square filtering aims to find a vectorized image $\vec{f}$ of a $N \times N$ image f that minimizes: $E(\vec{f}) = (L\vec{f})^T(L\vec{f})$ subject to the constraint: $[\vec{g} - H\vec{f}]^T[\vec{g} - H\vec{f}] = \epsilon$, for some block-circulant matrices H and L. ϵ is a fixed parameter greater than 0.

   (a) Let $W = W_2 \otimes W_2$, where $W_2(k,n) = \frac{1}{\sqrt{2}}e^{\pi jkn}$ for $0 \leq k,n \leq 1$ and $\otimes$ is the Kronecker product. Given that $H = W\Lambda_H W^{-1}$ and $L = W\Lambda_L W^{-1}$, where

   $$\Lambda_H = \begin{pmatrix} h_0 & 0 & 0 & 0 \\ 0 & h_1 & 0 & 0 \\ 0 & 0 & h_2 & 0 \\ 0 & 0 & 0 & h_3 \end{pmatrix} \text{ and } \Lambda_L = \begin{pmatrix} l_0 & 0 & 0 & 0 \\ 0 & l_1 & 0 & 0 \\ 0 & 0 & l_2 & 0 \\ 0 & 0 & 0 & l_3 \end{pmatrix}.$$

   where $h_i, l_i \in \mathbb{R}^+, 0 \leq i \leq 3$
   Let $\vec{g} = \mathcal{S}(g)$, where g is a 2×2 image and $\mathcal{S}$ is the stacking operator.

      i. Show that H is block-circulant
      ii. Show that $W^{-1}\mathcal{S}(h) = 2\mathcal{S}(\hat{h})$ for any 2×2 image h.

   (b) Show that the optimal solution $\vec{f} = \mathcal{S}(f)$ that solves the constrained least square problem satisfies $[\lambda H^T H + L^T L]\vec{f} = \lambda H^T \vec{g}$ for some parameter λ. Hence, find $DFT(f)$ in term of $DFT(g)$, $h_i, l_i, 0 \leq i \leq 3$ and λ. You may assume $\lambda > 0$. Please show your answer with details.