THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1010 UNIVERSITY MATHEMATICS 2023-2024 Term 1
Suggested Solutions of WeBWorK Coursework 3

If you find any errors or typos, please email us at
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1. (1 point) Evaluate the limit

$$\lim_{x\to 3}\left(\sqrt{x^2+2}-\frac{x^2+3x}{x}\right)$$

If the limit does not exist enter DNE.
Limit $= \sqrt{11}-6$

Solution:

$$\begin{aligned}\lim_{x\to 3}\left(\sqrt{x^2+2}-\frac{x^2+3x}{x}\right) &= \lim_{x\to 3}\sqrt{x^2+2}-\lim_{x\to 3}\frac{x^2+3x}{x}\\ &= \sqrt{11}-6\end{aligned}$$

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2. (1 point)
Let $f(x)=\begin{cases}\sqrt{-3-x}+2, & \text{if } x<-4\\ 2, & \text{if } x=-4\\ 3x+15, & \text{if } x>-4\end{cases}$

Calculate the following limits. Enter **DNE** if the limit does not exist.

$\lim_{x\to -4^-} f(x) = 3$

$\lim_{x\to -4^+} f(x) = 3$

$\lim_{x\to -4} f(x) = 3$

Solution:

(a)

$$\begin{aligned}\lim_{x\to -4^-} f(x) &= \lim_{x\to -4^-}(\sqrt{-3-x}+2)\\ &= 3\end{aligned}$$

(b)

$$\begin{aligned}\lim_{x\to -4^+} f(x) &= \lim_{x\to -4^+}(3x+15)\\ &= 3\end{aligned}$$

(c)

2 Because $\lim_{x\to -4^+} f(x) = \lim_{x\to -4^-} f(x)$, $\lim_{x\to -4} f(x)$ exists.

$$\lim_{x\to -4} f(x) = \lim_{x\to -4^+} f(x) = \lim_{x\to -4^-} f(x) = 3.$$

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3. (1 point) Evaluate the limits.

$$g(x) = \begin{cases} 6x+4 & x<10 \\ 60 & x=10 \\ 6x-4 & x>10 \end{cases}$$

Enter **DNE** if the limit does not exist.
a) $\lim_{x\to 10^-} g(x) = 64$
b) $\lim_{x\to 10^+} g(x) = 56$
c) $\lim_{x\to 10} g(x) =$ DNE
d) $g(10) = 60$

Solution:

(a)

$$\lim_{x\to -10^-} g(x) = \lim_{x\to 10^-} (6x+4)$$
$$= 64$$

(b)

$$\lim_{x\to 10^+} g(x) = \lim_{x\to 10^+} (6x-4)$$
$$= 56$$

(c)

$$\lim_{x\to 10^+} g(x) \neq \lim_{x\to 10^-} g(x)$$

So the limit does not exist.
(d)

$$g(10) = 60$$

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4. (1 point) Evaluate the limit

$$\lim_{x\to 2^-} \left(\frac{1}{x-2} - \frac{1}{|x-2|} \right)$$

Enter **INF** for ∞, **-INF** for $-\infty$, or **DNE** if the limit does not exist (i.e., there is no finite limit and neither ∞ nor $-\infty$ is the limit).

Limit $= -\infty$

Solution:

$$\lim_{x \to 2^-} \left(\frac{1}{x-2} - \frac{1}{|x-2|} \right) = \lim_{x \to 2^-} \left(\frac{1}{x-2} - \frac{1}{2-x} \right)$$
$$= \lim_{x \to 2^-} \left(\frac{2}{x-2} \right)$$
$$= -\infty$$

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5. (1 point) Use the Squeeze Theorem to evaluate the limit

$$\lim_{x \to 0} \sin x \cos \left(\frac{1}{x^2} \right)$$

Enter **DNE** if the limit does not exist.
Limit = 0

Solution:

Regardless of the value of $x \neq 0$,

$$-1 \leq \cos \left(\frac{1}{x^2} \right) \leq 1$$

Assume first that $x > 0$, and x is small enough so that $\sin x > 0$. Multiply the inequality by $\sin x$.

$$-\sin x \leq \sin x \cos \left(\frac{1}{x^2} \right) \leq \sin x$$

By the Squeeze Theorem, since $\lim_{x \to 0} (-\sin x) = \lim_{x \to 0} \sin x = 0$, we must have

$$\lim_{x \to 0} \sin x \cos \left(\frac{1}{x^2} \right) = 0.$$

The argument works in a similar way when $x < 0$ but close enough so that $\sin x < 0$.

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6. (1 point) Let

$$f(x) = \frac{x^2+1}{x^2-16}.$$

Find the indicated one-sided limits of f.
NOTE: Remember that you use **INF** for ∞ and **-INF** for $-\infty$.
You should also sketch a graph of $y = f(x)$, including vertical and horizontal asymptotes.
$\lim_{x \to -4^-} f(x) = \infty$
$\lim_{x \to -4^+} f(x) = -\infty$
$\lim_{x \to 4^-} f(x) = -\infty$
$\lim_{x \to 4^+} f(x) = \infty$
$\lim_{x \to -\infty} f(x) = 1$
$\lim_{x \to \infty} f(x) = 1$

4 **Solution:**

(a)

$$\lim_{x \to -4^-} f(x) = \lim_{x \to -4^-} \frac{x^2+1}{x^2-16} = \lim_{x \to -4^-} \frac{x^2+1}{(x-4)(x+4)}$$
$$= \infty$$

(b)

$$\lim_{x \to -4^+} f(x) = \lim_{x \to -4^+} \frac{x^2+1}{x^2-16} = \lim_{x \to -4^+} \frac{x^2+1}{(x-4)(x+4)}$$
$$= -\infty$$

(c)

$$\lim_{x \to 4^-} f(x) = \lim_{x \to 4^-} \frac{x^2+1}{x^2-16} = \lim_{x \to 4^-} \frac{x^2+16}{(x-4)(x+4)}$$
$$= -\infty$$

(d)

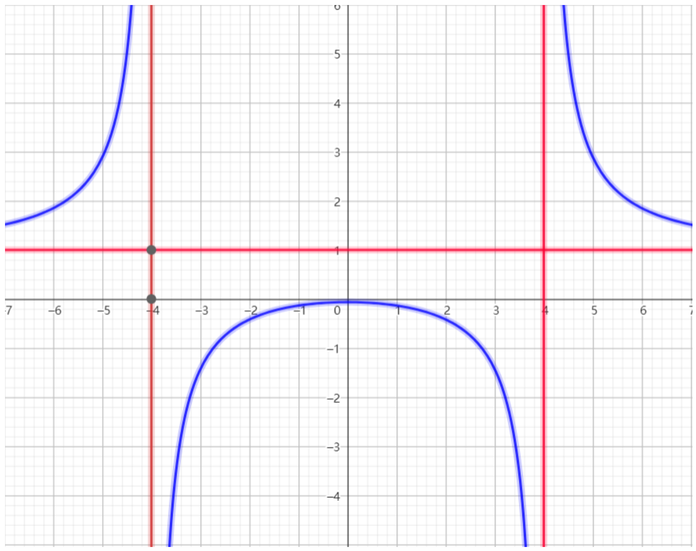
$$\lim_{x \to 4^+} f(x) = \lim_{x \to 4^+} \frac{x^2+1}{x^2-16} = \lim_{x \to 4^+} \frac{x^2+1}{(x-4)(x+4)}$$
$$= \infty$$

(e)

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \frac{x^2+1}{x^2-16} = \lim_{x \to -\infty} \frac{1+\frac{1}{x^2}}{1-\frac{16}{x^2}}$$
$$= 1$$

(f)

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{x^2+1}{x^2-16} = \lim_{x \to \infty} \frac{1+\frac{1}{x^2}}{1-\frac{16}{x^2}}$$
$$= 1$$



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7. (1 point)

a. [choose/true/false] If $\lim_{x \to 1^-} f(x) = 5$, then $\lim_{x \to 1} f(x) = 5$.

b. [choose/true/false] If $\lim_{x \to 1^-} f(x) = 5$, then $\lim_{x \to 1^+} f(x) = 5$.

c. [choose/true/false] If $\lim_{x \to 1} f(x) = 5$, then $\lim_{x \to 1^-} f(x) = 5$.

d. [choose/true/false] If $\lim_{x \to 1} f(x) = 5$, then $\lim_{x \to 1^+} f(x) = 5$.

e. Select all true statements. Assume that all the limits are all taken at the same point.

- A. If the left- and right-hand limits both exist and are equal, then the two-sided limit exists.
- B. If the left-hand limit exists, then the two-sided limit exists.
- C. If the two-sided limit exists, then the left- and right-hand limits both exist and are equal.
- D. If the right-hand limit exists, then the two-sided limit exists.

Solution:
(a.) False
(b.) False
(c.) True
(d.) True
(e.) AC

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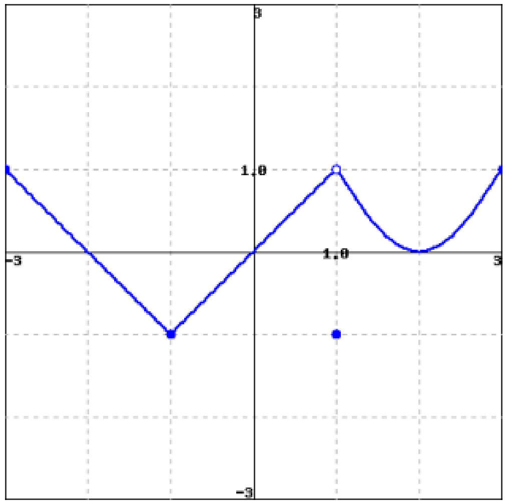
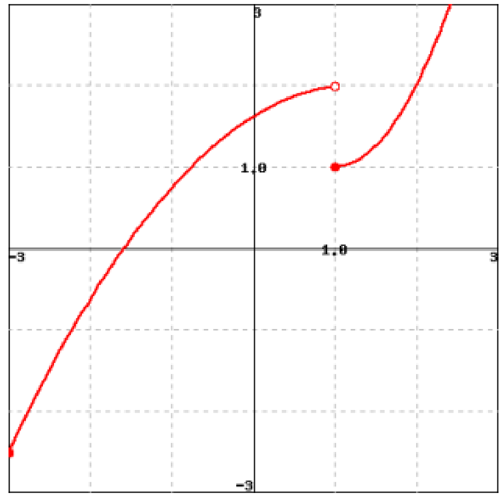
8. (1 point) Use the given graphs of the function f (left, in blue) and g (right, in red) to find the following limits:





1. $\lim_{x\to 1}[f(x)+g(x)] = \text{DNE}$ help (limits)

2. $\lim_{x\to 2}[f(x)+g(x)] = 2$

3. $\lim_{x\to 0} f(x)g(x) = 0$

4. $\lim_{x\to 0} \dfrac{f(x)}{g(x)} = 0$

5. $\lim_{x\to -1} \sqrt{3+f(x)} = \sqrt{2}$

Note: *You can click on the graphs to enlarge the images.*

Solution:

1. When x appoarching 1 from two sides, left and right limit of $g(x)$ are not equal, however in this case limit of $f(x)$ exists, so limit of $f(x)+g(x)$ does not exist.

2. Limit exist this time for both $f(x)$ and $g(x)$, simply add up these two limits we get 2.

3. Again limit of $f(x)$ and $g(x)$ exist, as $\lim_{x\to 0} f(x) = 0$, the answer should be 0.

4. Notice limit of $g(x)$ exists and it is a nonzero real number, so limit of $\frac{f(x)}{g(x)}$ is totally controlled by the behavior of $f(x)$ near 0. Since around 0 you can invert $g(x)$ locally, i.e. $\frac{f(x)}{g(x)} = \frac{1}{g(x)} f(x)$, similar to 8.3 we get 0.

5. Routinely check that $\lim_{x\to -1} f(x)$ exists, so does $\lim_{x\to -1} \sqrt{3+f(x)}$, since $\lim_{x\to -1} f(x) = -1$, the answer should be $\sqrt{2}$.

9. (1 point)

Find $\lim\limits_{x\to 0^+} \sqrt{x}\, e^{\sin(\pi/x)}$

0

Solution:

Although $e^{\sin\frac{\pi}{x}}$ looks awful, it's actually bounded by positive real numbers e^{-1} and e as $\sin\frac{\pi}{x}$ is bounded by -1 and 1. So the really effective part is $\sqrt{x}$, as $\lim\limits_{x\to 0^+} \sqrt{x} = 0$ (Since square root only defined on non negative real number!), the answer should be 0.

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10. (1 point)

Evaluate the limit:

$$\lim_{x\to 0} \frac{x^2}{\sin^2(3x)} = \frac{1}{9}$$

Solution:

$$\lim_{x\to 0} \frac{x^2}{\sin^2(3x)} = \lim_{x\to 0} \frac{1}{\frac{\sin(3x)}{x} \cdot \frac{\sin(3x)}{x}} = \lim_{x\to 0} \frac{1}{\frac{3\sin(3x)}{3x}} \cdot \lim_{x\to 0} \frac{1}{\frac{3\sin(3x)}{3x}} = \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}.$$

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11. (1 point)

Evaluate the limit:

$$\lim_{x\to 0} \frac{\tan 5x}{\tan 8x} = \frac{5}{8}$$

Solution:

$$\lim_{x\to 0} \frac{\tan 5x}{\tan 8x} = \lim_{x\to 0} \frac{\cos 8x}{\cos 5x} \cdot \frac{\sin 5x}{5x} \cdot \frac{5}{8} \cdot \frac{8x}{\sin 8x} = \frac{5}{8}.$$