THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1010 University Mathematics 2023-2024 Term 1
Suggested Solutions of Homework Assignment 1

1. Determine the limit of each of the following sequences, or show that the sequence diverges. You may make use of the limit laws and theorems covered in class.

(a) $a_n = \dfrac{3n-5}{n+1} - \left(\dfrac{3}{5}\right)^n$ for $n \geq 1$.

(b) $a_n = \sqrt{n}(\sqrt{n+5} - \sqrt{n})$ for $n \geq 1$.

(c) $a_n = \dfrac{3^n}{n!}$ for $n \geq 1$.

(d) $a_n = \dfrac{\sin(n^2)}{n}$ for $n \geq 1$.

(e) $a_n = \dfrac{n}{n + n^{1/n}}$ for $n \geq 1$.

(f) $a_n = \left(5 + \dfrac{4}{n^2}\right)^{1/3}$ for $n \geq 1$.

Solution:

(a)

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left[\frac{3n-5}{n+1} - \left(\frac{3}{5}\right)^n\right] = \lim_{n\to\infty}\left[\frac{3-\frac{5}{n}}{1+\frac{1}{n}} - \left(\frac{3}{5}\right)^n\right] = \frac{3-0}{1+0} - 0 = \boxed{3}$$

(b)

$$\begin{aligned}\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \sqrt{n}\left(\sqrt{n+5} - \sqrt{n}\right) \cdot \frac{\sqrt{n+5}+\sqrt{n}}{\sqrt{n+5}+\sqrt{n}} \\ &= \lim_{n\to\infty} \frac{\sqrt{n}\cdot(n+5-n)}{\sqrt{n+5}+\sqrt{n}} \\ &= \lim_{n\to\infty} \frac{1\cdot 5}{\sqrt{1+\dfrac{5}{n}}+1} = \frac{5}{\sqrt{1+0}+1} = \boxed{\frac{5}{2}}\end{aligned}$$

(c) Note that for $n > 3$,

$$a_n = \frac{3^3}{3!}\cdot\frac{3}{4}\cdot\frac{3}{5}\cdot \ldots \cdot\frac{3}{n} < \frac{3^3}{3!}\cdot 1 \cdot\ 1 \cdot\ \ldots \cdot\ \frac{3}{n} = \frac{3^4}{3!}\cdot\frac{1}{n}$$

Then for $n > 3$, we have

$$0 < a_n < \frac{3^4}{3!}\cdot\frac{1}{n}$$

Since $\lim\limits_{n\to\infty} \dfrac{3^4}{3!}\cdot\dfrac{1}{n} = 0$, by squeeze theorem, $\lim\limits_{n\to\infty} a_n = \boxed{0}$.

(d) We have $-1 \leq \sin n^2 \leq 1$ and so $\dfrac{-1}{n} \leq \dfrac{\sin n^2}{n} \leq \dfrac{1}{n}$.

Since $\lim_{n\to\infty} \dfrac{-1}{n} = 0$ and $\lim_{n\to\infty} \dfrac{1}{n} = 0$, by squeeze theorem, $\lim_{n\to\infty} a_n = \boxed{0}$.

(e) (Method 1)

We first prove that $0 < n^{1/n} < 2$.

Clearly, $n^{1/n} > 0$ since n is positive.

We can use mathematical induction to prove that $n < 2^n$, hence $n^{1/n} < 2$.

For $n = 1, 2^1 = 2 > 1$.

Assume the statement is true for $n = k$, i.e. $k < 2^k$.

Then, for $n = k+1, k+1 \leq 2k < 2 \cdot 2^k = 2^{k+1}$.

Therefore, we have $0 < n^{1/n} < 2$.

Hence,

$$\frac{n}{n+2} < \frac{n}{n+n^{1/n}} < \frac{n}{n+0} = 1.$$

Since $\lim_{n\to\infty} \dfrac{n}{n+2} = 1$, by squeeze theorem, $\lim_{n\to\infty} a_n = \boxed{1}$.

(Method 2)

Another way to find the limit is as follows:

$$\lim_{n\to\infty} \frac{n}{n+n^{1/n}} = \lim_{n\to\infty} \frac{1}{1+n^{1/n-1}} = \lim_{n\to\infty} \frac{1}{1+\left(\frac{1}{n}\right)^{1-1/n}} = \frac{1}{1+0^{1-0}} = \boxed{1}.$$

(f)

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(5 + \frac{4}{n^2}\right)^{1/3} = (5+0)^{1/3} = \boxed{5^{1/3}}$$

2. Consider the following bounded and increasing sequence:

$$\begin{cases} a_1 = \sqrt{3} \\ a_2 = \sqrt{3+\sqrt{3}} \\ a_3 = \sqrt{3+\sqrt{3+\sqrt{3}}} \\ \vdots \\ a_{n+1} = \sqrt{3+a_n} \\ \vdots \end{cases}$$

Answer the following questions:

(a) Show that the sequence converges and find its limit.

(b) Answer the same question when 3 is replaced by an arbitrary integer $k \geq 2$.

Solution:

(a) (i) Let $P(n)$ be the statement that $a_{n+1} \geq a_n$.

- When $n = 1$,

$$a_2 = \sqrt{3+\sqrt{3}} > \sqrt{3} = a_1$$

Hence, $P(1)$ is true.

- Suppose $P(m)$ is true, i.e.

$$a_{m+1} \geq a_m$$

- When $n = m+1$,

$$a_{m+2} = \sqrt{3+a_{m+1}} \geq \sqrt{3+a_m} = a_{m+1}$$

Hence, $P(m+1)$ is true.

Therefore, $P(n)$ is true for any $n \geq 1$, i.e. $\{a_n\}$ is increasing.

(ii) Let $Q(n)$ be the statement that $a_{n+1} \leq \frac{1+\sqrt{13}}{2}$.

- When $n = 1$,

$$a_1 = \sqrt{3} < \sqrt{\frac{13}{4}} = \frac{\sqrt{13}}{2} < \frac{1+\sqrt{13}}{2}$$

Hence, $Q(1)$ is true.

- Suppose $Q(m)$ is true, i.e.

$$a_m \leq \frac{1+\sqrt{13}}{2}$$

- When $n = m+1$,

$$a_{m+1} = \sqrt{3+a_m} \leq \sqrt{3+\frac{1+\sqrt{13}}{2}} = \frac{\sqrt{1+2\sqrt{13}+13}}{2} = \frac{1+\sqrt{13}}{2}$$

Hence, $Q(m+1)$ is true.

Therefore, $Q(n)$ is true for any $n \geq 1$, i.e. $a_n \leq \frac{1+\sqrt{13}}{2}$.

By Monotone Convergence Theorem, $\{a_n\}$ is convergent.

Suppose $\lim_{n\to\infty} a_n = L$.

$$a_{n+1} = \sqrt{3+a_n}$$
$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{3+a_n}$$
$$L = \sqrt{3+L}$$
$$L^2 - L - 3 = 0$$
$$L = \frac{1+\sqrt{13}}{2} \quad \text{or} \quad L = \frac{1-\sqrt{13}}{2}$$

$L = \frac{1-\sqrt{13}}{2}$ is rejected since $a_n > 0$ for all n. Hence, $\lim_{n\to\infty} a_n = \boxed{\frac{1+\sqrt{13}}{2}}$.

(b) For any integer $k \geq 2$,

(i) Let $P(n)$ be the statement that $a_{n+1} \geq a_n$.

- When $n = 1$,

$$a_2 = \sqrt{k + \sqrt{k}} > \sqrt{k} = a_1$$

Hence, $P(1)$ is true.

- Suppose $P(m)$ is true, i.e.

$$a_{m+1} \geq a_m$$

- When $n = m + 1$,

$$a_{m+2} = \sqrt{k + a_{m+1}} \geq \sqrt{k + a_m} = a_{m+1}$$

Hence, $P(m + 1)$ is true.

Therefore, $P(n)$ is true for any $n \geq 1$, i.e. $\{a_n\}$ is increasing.

(ii) Let $Q(n)$ be the statement that $a_{n+1} \leq \frac{1+\sqrt{1+4k}}{2}$.

- When $n = 1$,

$$a_1 = \sqrt{k} < \sqrt{\frac{1+4k}{4}} = \frac{\sqrt{1+4k}}{2} < \frac{1+\sqrt{1+4k}}{2}$$

Hence, $Q(1)$ is true.

- Suppose $Q(m)$ is true, i.e.

$$a_m \leq \frac{1+\sqrt{1+4k}}{2}$$

- When $n = m + 1$,

$$\begin{aligned} a_{m+1} = \sqrt{k + a_m} &\leq \sqrt{k + \frac{1+\sqrt{1+4k}}{2}} \\ &= \frac{\sqrt{1 + 2\sqrt{1+4k} + 1 + 4k}}{2} = \frac{1+\sqrt{1+4k}}{2} \end{aligned}$$

Hence, $Q(m + 1)$ is true.

Therefore, $Q(n)$ is true for any $n \geq 1$, i.e. $a_n \leq \frac{1+\sqrt{1+4k}}{2}$.

By Monotone Convergence Theorem, $\{a_n\}$ is convergent.
Suppose $\lim_{n\to\infty} a_n = L$.

$$\begin{aligned} a_{n+1} &= \sqrt{k + a_n} \\ \lim_{n\to\infty} a_{n+1} &= \lim_{n\to\infty} \sqrt{k + a_n} \\ L &= \sqrt{k + L} \\ L^2 - L - k &= 0 \end{aligned}$$

$$L = \frac{1+\sqrt{1+4k}}{2} \quad \text{or} \quad L = \frac{1-\sqrt{1+4k}}{2}$$

$L = \frac{1-\sqrt{1+4k}}{2}$ is rejected since $a_n > 0$ for all n. Hence, $\lim_{n\to\infty} a_n = \boxed{\frac{1+\sqrt{1+4k}}{2}}$.

3. For this problem, you may make use of the following mathematical result:

Fact. Let a, r be real numbers, with $r \neq 1$. Let $\{S_n\}$ be the geometric series defined as follows:

$$S_n = \sum_{k=0}^{n} ar^k = a + ar + ar^2 + \cdots + ar^n, \quad n = 0, 1, 2, \ldots.$$

Then, $S_n = a\left(\dfrac{1 - r^{n+1}}{1 - r}\right)$.

(a) Verify that $\{S_n\}$ converges to $\dfrac{a}{1-r}$, whenever $|r| < 1$.

(b) Use the result of Part (a) to find the limit of the sequence $\{a_n\}$, where

$$a_n = 1 + \frac{3}{4} + \frac{3}{4^2} + \cdots + \frac{3}{4^n}.$$

(c) Use the result of Part (a) to verify that the repeating decimal $1.777\cdots$, often written as $1.\dot{7}$, is equal to $\dfrac{16}{9}$.

Solution:

(a) When $|r| < 1$, we have $1 - r \neq 0$ and $\lim_{n\to\infty} r^{n+1} = 0$. Then

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} a\left(\frac{1 - r^{n+1}}{1 - r}\right) = a\left(\frac{1 - \lim_{n\to\infty} r^{n+1}}{1 - r}\right) = a\left(\frac{1 - 0}{1 - r}\right) = \frac{a}{1 - r}.$$

(b) Let $a = 3$ and $r = \dfrac{1}{4}$. Then $a_n = S_n - 2$.

Then $\lim_{n\to\infty} a_n = \lim_{n\to\infty} S_n - 2 = \dfrac{a}{1-r} - 2 = \dfrac{3}{1 - \frac{1}{4}} - 2 = \boxed{2}$.

(c) Let $a = 7$ and $r = \dfrac{1}{10}$. Then $a_n = S_n - 6$.

Then $1.\dot{7} = \lim_{n\to\infty} a_n = \lim_{n\to\infty} S_n - 6 = \dfrac{a}{1-r} - 6 = \dfrac{7}{1 - \frac{1}{10}} - 6 = \dfrac{16}{9}$.

4. A sequence $\{a_n\}$ is defined recursively by the following equations:

$$\begin{cases} a_1 = 1, \\ a_{n+1} = \sqrt{7 + 2a_n} \quad \text{for } n \geq 1. \end{cases}$$

Answer the following questions:

(a) Show that $\{a_n\}$ is bounded and monotonic and hence convergent.

(b) Find the limit of $\{a_n\}$.

Solution:

(a) (i) Let $P(n)$ be the statement that $a_{n+1} \geq a_n$.

- When $n = 1$,
$$a_2 = \sqrt{7+2} = 3 > 1 = a_1$$
Hence, $P(1)$ is true.
- Suppose $P(m)$ is true, i.e.
$$a_{m+1} \geq a_m$$
- When $n = m+1$,
$$a_{m+2} = \sqrt{7 + 2a_{m+1}} \geq \sqrt{7 + 2a_m} = a_{m+1}$$
Hence, $P(m+1)$ is true.

Therefore, $P(n)$ is true for any $n \geq 1$, i.e. $\{a_n\}$ is increasing.

(ii) Let $Q(n)$ be the statement that $a_{n+1} \leq 1 + 2\sqrt{2}$.

- When $n = 1$,
$$a_1 = 1 < 1 + 2\sqrt{2}$$
Hence, $Q(1)$ is true.
- Suppose $Q(m)$ is true, i.e.
$$a_m \leq 1 + 2\sqrt{2}$$
- When $n = m+1$,
$$a_{m+1} = \sqrt{7 + 2a_m} \leq \sqrt{7 + 2 + 4\sqrt{2}} = \sqrt{1 + 2 \times 2\sqrt{2} + 8} = 1 + 2\sqrt{2}$$
Hence, $Q(m+1)$ is true.

Therefore, $Q(n)$ is true for any $n \geq 1$, i.e. $a_n \leq 1 + 2\sqrt{2}$.

By Monotone Convergence Theorem, $\{a_n\}$ is convergent.

(b) Suppose $\lim_{n\to\infty} a_n = L$.

$$a_{n+1} = \sqrt{7 + 2a_n}$$
$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{7 + 2a_n}$$
$$L = \sqrt{7 + 2L}$$
$$L^2 - 2L - 7 = 0$$
$$L = 1 + 2\sqrt{2} \quad \text{or} \quad L = 1 - 2\sqrt{2}$$

$L = 1 - 2\sqrt{2}$ is rejected since $a_n > 0$ for all n. Hence, $\lim_{n\to\infty} a_n = \boxed{1 + 2\sqrt{2}}$.

5. Let $k > 0$ and a_1 be a positive number. Define a sequence $\{a_n\}$ by the relation:

$$a_{n+1} = \sqrt{k + a_n} \quad \text{for } n \geq 1.$$

Let α be the positive root of the equation:

$$x^2 - x - k = 0.$$

(a) Suppose $0 < a_1 < \alpha$. Show that the sequence $\{a_n\}$ is monotonic increasing and converges to α.

(b) Suppose $a_1 > \alpha$. Show that the sequence $\{a_n\}$ is monotonic decreasing and converges to α.

Solution:

(a) Let $P(n)$ be the statement that $a_{n+1} \geq a_n$.

- First we note that $x^2 - x - k = 0$ has a positive root α and a negative root $-k/\alpha$, and that $x^2 - x - k < 0$ whenever $-k/\alpha < x < \alpha$.
  Since $0 < a_1 < \alpha$, we have $a_1^2 - a_1 - k < 0$, and so $a_1 < \sqrt{k + a_1} = a_2$.
  Hence, $P(1)$ is true.
- Suppose $P(m)$ is true, i.e. $a_{m+1} \geq a_m$.
- When $n = m + 1$,

$$a_{m+2} = \sqrt{k + a_{m+1}} \geq \sqrt{k + a_m} = a_{m+1}.$$

  Hence, $P(m + 1)$ is true.

By mathematical induction, $P(n)$ is true for all $n \geq 1$, i.e. $\{a_n\}$ is monotonic increasing.

Next, we show that $\{a_n\}$ is bounded above by α. Let $Q(n)$ be the statement that $a_n < \alpha$.

- Clearly, $a_1 < \alpha$. Hence, $Q(1)$ is true.
- Suppose $Q(m)$ is true, i.e. $a_m < \alpha$.
- When $n = m + 1$,

$$a_{m+1} = \sqrt{k + a_m} < \sqrt{k + \alpha} = \sqrt{\alpha^2} = \alpha.$$

  Hence, $Q(m + 1)$ is true.

By mathematical induction, $Q(n)$ is true for all $n \geq 1$. So $\{a_n\}$ is bounded above by α.

By Monotone Convergence Theorem, $\{a_n\}$ converges. Let $\ell = \lim_{n \to +\infty} a_n$. Then

$$\begin{aligned} \lim_{n \to +\infty} a_{n+1}^2 &= \lim_{n \to +\infty} (k + a_n) \\ \ell^2 - \ell - k &= 0. \end{aligned}$$

Since $a_n \geq a_1 > 0$ for all $n \geq 1$, we have $\ell \geq a_1 > 0$.

So ℓ is the positive root of $x^2 - x - k = 0$. Therefore, $\lim_{n \to +\infty} a_n = \ell = \alpha$.

(b) Let $P(n)$ be the statement that $a_{n+1} < a_n$ and $a_n > \alpha$.

- Since $a_1 > \alpha$, we have $a_1^2 - a_1 - k > 0$, and so $a_1 > \sqrt{k + a_1} = a_2$. Hence, $P(1)$ is true.
- Suppose $P(m)$ is true, i.e. $a_{m+1} < a_m$ and $a_m > \alpha$..
- When $n = m + 1$,

$$a_{m+2} = \sqrt{k + a_{m+1}} < \sqrt{k + a_m} = a_{m+1},$$

and

$$a_{m+1} = \sqrt{k + a_m} > \sqrt{k + \alpha} = \sqrt{\alpha^2} = \alpha.$$

Hence, $P(m + 1)$ is true.

By mathematical induction, $P(n)$ is true for all $n \geq 1$. Thus, $\{a_n\}$ is monotonic decreasing and bounded below by α.

By Monotone Convergence Theorem, $\{a_n\}$ converges. Let $\ell = \lim_{n\to+\infty} a_n$. Then

$$\begin{aligned}\lim_{n\to+\infty} a_{n+1}^2 &= \lim_{n\to+\infty} (k + a_n)\\ \ell^2 - \ell - k &= 0.\end{aligned}$$

Since $a_n > \alpha > 0$ for all $n \geq 1$, we have $\ell \geq \alpha > 0$.

So ℓ is the positive root of $x^2 - x - k = 0$. Therefore, $\lim_{n\to+\infty} a_n = \ell = \alpha$.

6. Given a sequence $\{a_n\}$ such that $a_1 > a_2 > 0$, and

$$a_{n+2} = \frac{1}{2}(a_{n+1} + a_n), \quad \text{for} \quad n = 1, 2, \cdots.$$

Answer the following questions:

(a) Show that for $n \geq 1$,

$$a_{n+2} - a_n = \frac{(-1)^n}{2^n}(a_1 - a_2)$$

and hence show that the sequence $\{a_1, a_3, a_5, \cdots\}$ is strictly decreasing and that the sequence $\{a_2, a_4, a_6, \cdots\}$ is strictly increasing.

(b) For any positive integers m and n, show that

$$a_{2m} < a_{2n-1}.$$

(c) Show that the two sequences $\{a_1, a_3, a_5, \cdots\}$ and $\{a_2, a_4, a_6, \cdots\}$ converge to the same limit k, where

$$k = \frac{1}{3}(a_1 + 2a_2).$$

Solution:

(a) Because

$$a_{n+1} - a_n = \frac{1}{2}(a_n + a_{n-1}) - a_n = -\frac{1}{2}(a_n - a_{n-1}),$$

we have

$$\begin{aligned} a_{n+1} - a_n &= -\frac{1}{2}(a_n - a_{n-1}) \\ &= \left(-\frac{1}{2}\right)^2 (a_{n-1} - a_{n-2}) \\ &= \left(-\frac{1}{2}\right)^3 (a_{n-2} - a_{n-3}) \\ &\;\vdots \\ &= \left(-\frac{1}{2}\right)^{n-1} (a_2 - a_1). \end{aligned}$$

Hence,

$$\begin{aligned} a_{n+2} - a_n &= \frac{1}{2}(a_{n+1} + a_n) - a_n \\ &= \frac{1}{2}(a_{n+1} - a_n) \\ &= \frac{1}{2}\left(-\frac{1}{2}\right)^{n-1} (a_2 - a_1) \\ &= \left(-\frac{1}{2}\right)^{n} (a_1 - a_2). \end{aligned}$$

Since $a_1 - a_2 > 0$, it follows that $a_{n+2} - a_n \begin{cases} > 0 & \text{when } n \text{ is even} \\ < 0 & \text{when } n \text{ is odd} \end{cases}$.

Accordingly, $\{a_{2n+1}\}$ is strictly decreasing and $\{a_{2n}\}$ is strictly increasing.

(b) For any $m, n \geq 1$, consider the following 3 cases:

(i) Let $m = n$. By (a), $2a_{2m} = a_{2m-1} + a_{2m-2} < a_{2m-1} + a_{2m}$. So $a_{2m} < a_{2m-1}$.

(ii) Let $m < n$. By (a) and (b)(i), $a_{2m} < a_{2n} < a_{2n-1}$.

(iii) Let $m > n$. By (a) and (b)(i), $a_{2n-1} > a_{2m-1} > a_{2m}$.

In all cases, $a_{2m} < a_{2n-1}$ for $m, n \geq 1$.

(c) By (a) and (b), $\{a_{2n+1}\}$ is decreasing and bounded below, e.g. by a_2, $\{a_{2n}\}$ is increasing and bounded above, e.g. by a_1. So, by Monotone Convergence Theorem, both sequences converge. Let $\lim_{n\to\infty} a_{2n} = \ell_1$ and $\lim_{n\to\infty} a_{2n+1} = \ell_2$. Then $\lim_{n\to\infty} a_{n+2} = \lim_{n\to\infty} \frac{1}{2}(a_{n+1} + a_n)$ implies that

$$\begin{cases} \ell_2 = \frac{1}{2}(\ell_1 + \ell_2) & \text{if } n \text{ is odd} \\ \ell_1 = \frac{1}{2}(\ell_2 + \ell_1) & \text{if } n \text{ is even}. \end{cases}$$

Thus, $\ell_1 = \ell_2$, i.e. $\lim_{n\to\infty} a_{2n} = \lim_{n\to\infty} a_{2n+1}$.
Now, from the definition of the sequence,

$$\begin{aligned}\sum_{k=3}^{n} a_k &= \frac{1}{2}\sum_{k=3}^{n}(a_{k-2}+a_{k-1}) \\ &= \frac{1}{2}a_1 + \sum_{k=2}^{n-2} a_k + \frac{1}{2}a_{n-1} \\ \frac{1}{2}a_{n-1} + a_n &= \frac{1}{2}a_1 + a_2.\end{aligned}$$

Taking limit,

$$\begin{aligned}\frac{3}{2}\lim_{n\to\infty} a_n &= \frac{1}{2}a_1 + a_2 \\ \lim_{n\to\infty} a_n &= \frac{1}{3}(a_1 + 2a_2).\end{aligned}$$

7. For each of the given functions, f, find its natural domain, that is, the largest subset of $\mathbb{R}$ on which the expression defining f may be validly computed. Please express your answer in the form of a single interval, or a union of disjoint intervals. For example: $(-\infty, 2) \cup [5, 11)$.

   (a) (Optional) $f(x) = \dfrac{1}{2}\sqrt{4-x^2}$.

   (b) $f(x) = \sqrt{\dfrac{x-3}{x+3}}$.

   (c) (Optional) $f(x) = \ln(3x^2 - 4x + 5)$.

   (d) $f(x) = \ln(\sqrt{x-4} + \sqrt{6-x})$.

   (e) (Optional) $f(x) = \sin^2 x + \cos^4 x$.

   (f) $f(x) = \dfrac{1}{1+\cos x}$.

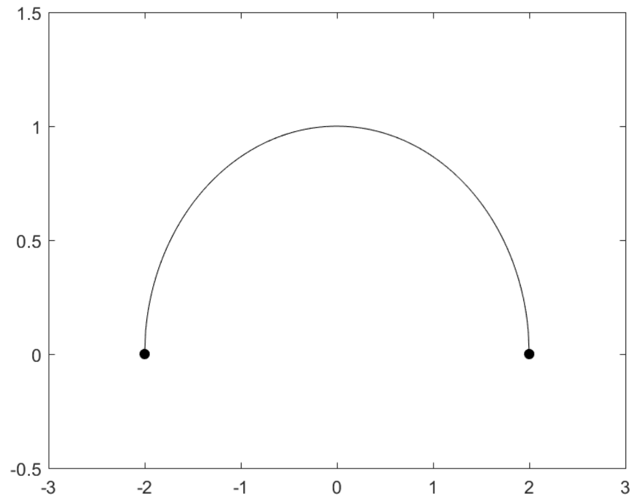
   (g) $f(x) = 1 - |x-1|$.

Solution:

(a)

$$f(x) = \frac{1}{2}\sqrt{4-x^2}$$

It implies the condition $4 - x^2 \geq 0, -2 \leq x \leq 2$.
Hence, the largest domain is $\boxed{[-2, 2]}$.



(b)

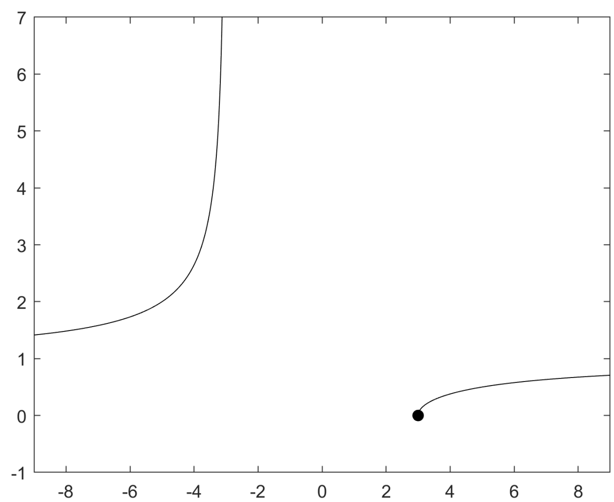
$$f(x) = \sqrt{\frac{x-3}{x+3}}$$

It implies two conditions $x \neq -3$ and $\dfrac{x-3}{x+3} \geq 0$.

For $\dfrac{x-3}{x+3} \geq 0$,

$$\begin{aligned} \frac{x-3}{x+3} &\geq 0 \\ \frac{x-3}{x+3} \cdot (x+3)^2 &\geq 0 \\ (x-3)(x+3) &\geq 0 \\ x \leq -3 &\text{ or } x \geq 3 \end{aligned}$$

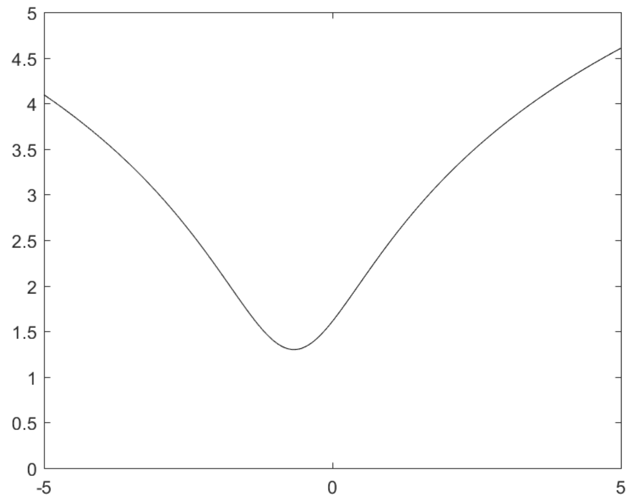
Hence, the largest domain is $\boxed{(-\infty, -3) \cup [3, \infty)}$.



(c)

$$f(x) = \ln(3x^2 - 4x + 5)$$

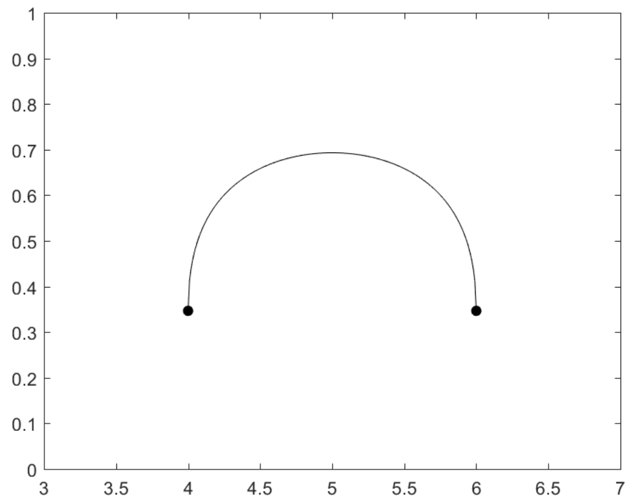
It implies the condition $3x^2 - 4x + 5 > 0$.
Note that $\Delta = (-4)^2 - 4 \cdot 3 \cdot 5 = -44 < 0$, so the equation has no real roots.
Then $3x^2 - 4x + 5 > 0$ for any x.
Hence, the largest domain is $\boxed{(-\infty, \infty)}$.



(d)

$$f(x) = \ln(\sqrt{x-4} + \sqrt{6-x})$$

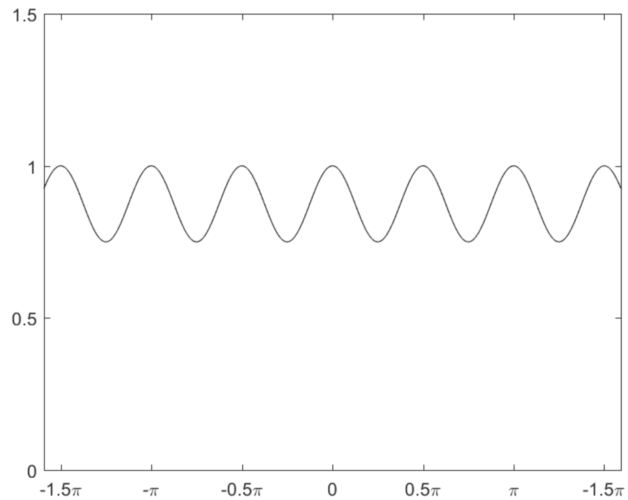
It implies three conditions $x - 4 \geq 0$, $6 - x \geq 0$, and $\sqrt{x-4} + \sqrt{6-x} > 0$.
We get $4 \leq x \leq 6$ from the first two conditions.
For the third condition, note that $\sqrt{x-4} \geq 0$ and $\sqrt{6-x} \geq 0$, and they cannot be 0 simultaneously, so any number satisfying $4 \leq x \leq 6$ works.
Hence, the largest domain is $\boxed{[4, 6]}$.



(e)

$$f(x) = \sin^2 x + \cos^4 x$$

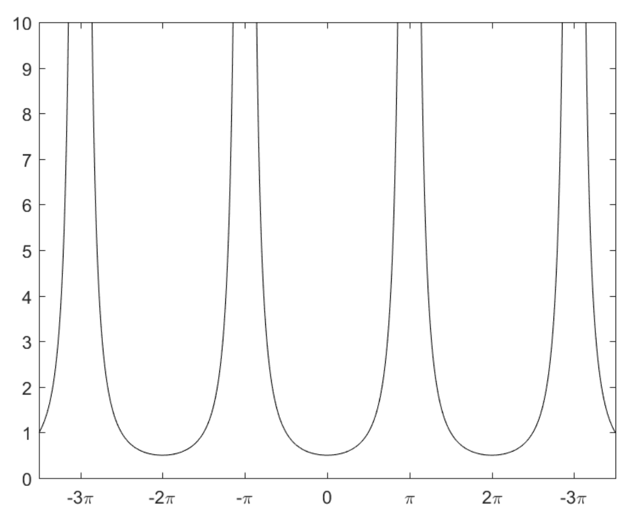
Note that $\sin x$ and $\cos x$ do not impose any conditions on domain.
Hence, the largest domain is $\boxed{(-\infty, \infty)}$.



(f)

$$f(x) = \frac{1}{1 + \cos x}$$

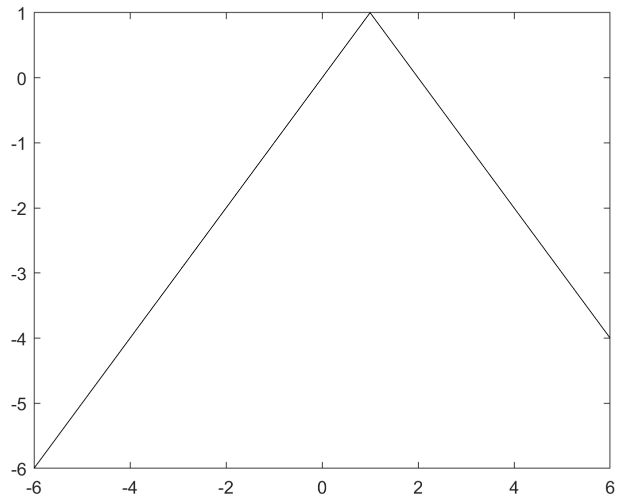
It implies the condition $\cos x \neq -1$.
Therefore, we have $x \neq \pi + 2n\pi$, where n is any integer.
To write the largest domain in disjoint interval, it involves infinitely many intervals of the form $((2n + 1)\pi, (2n + 3)\pi)$
We can write it as $\boxed{\bigcup_{n \in \mathbb{Z}} ((2n + 1)\pi, (2n + 3)\pi)}$.



(g)

$$f(x) = 1 - |x - 1|$$

Note that $|x-1|$ do not impose any conditions on domain.
Hence, the largest domain is $\boxed{(-\infty, \infty)}$.



8. Determine whether the given function, f, is injective, surjective, bijective, or none of these. Explain clearly.

   (a) $f : \mathbb{R} \to \mathbb{R}$, where $f(x) = 2x - 1$.

   (b) $f : \{x |\ x \neq 1\} \to \mathbb{R}$, where $f(x) = \dfrac{x^2 - 1}{x - 1}$.

   (c) $f : \mathbb{R} \to \mathbb{R}$, where $f(x) = \sqrt[3]{x}$.

   (d) $f : [-1, 1] \to [0, 4)$, where $f(x) = x^2$.

Solution:

(a) For any $x_1, x_2 \in \mathbb{R}$ with $x_1 \neq x_2$, we have $f(x_1) = 2x_1 - 1 \neq f(x_2) = 2x_2 - 1$. Therefore, f is $\boxed{\text{injective}}$.
For any $y \in \mathbb{R}$, there exists $x = \frac{y+1}{2} \in \mathbb{R}$ such that $f(x) = 2x - 1 = 2(\frac{y+1}{2}) = y$. Therefore, f is $\boxed{\text{surjective}}$.
Since f is both injective and surjective, it is $\boxed{\text{bijective}}$.

(b) Note that for $x \in (-\infty, 1) \cup (1, +\infty)$, $f(x) = \dfrac{x^2 - 1}{x - 1} = x + 1$.
For any $x_1, x_2 \in (-\infty, 1) \cup (1, +\infty)$ with $x_1 \neq x_2$, we have $f(x_1) = x_1 + 1 \neq f(x_2) = x_2 + 1$. Therefore, f is $\boxed{\text{injective}}$.
For $y = 2 \in \mathbb{R}$, there exists no $x \in (-\infty, 1) \cup (1, +\infty)$ such that $f(x) = y$ (otherwise, $x^2 - 1 = 2(x - 1) \implies (x - 1)^2 = 0 \implies x = 1$, which is a

contradiction). Therefore, f is $\boxed{\text{not surjective}}$.
As f is not surjective, it is $\boxed{\text{not bijective}}$.

(c) For any $x_1, x_2 \in \mathbb{R}$ with $x_1 \neq x_2$, we have $f(x_1) = \sqrt[3]{x_1} \neq f(x_2) = \sqrt[3]{x_2}$. Then f is $\boxed{\text{injective}}$.
For any $y \in \mathbb{R}$, there exists $x = y^3 \in \mathbb{R}$ such that $f(x) = \sqrt[3]{x} = \sqrt[3]{y^3} = y$. Therefore, f is $\boxed{\text{surjective}}$.
Since f is both injective and surjective, it is $\boxed{\text{bijective}}$.

(d) Note that we have $-1 \neq 1$ but $f(-1) = (-1)^2 = 1$ and $f(1) = 1^2 = 1$. Therefore, f is $\boxed{\text{not injective}}$.
For $y = 2 \in [0, 4)$, there exists no $x \in [-1, 1]$ such that $f(x) = y$ (since $x^2 = 2 \Leftrightarrow x = \pm\sqrt{2}$ which are outside $[-1, 1]$). Therefore, $f(x)$ is $\boxed{\text{not surjective}}$.
As f is not injective, it is $\boxed{\text{not bijective}}$.

9. Determine whether the given function, f, is increasing, strictly increasing, decreasing, strictly decreasing, bounded, bounded above, or bounded below.

(a) $f : [0, +\infty) \to \mathbb{R}$, where $f(x) = \dfrac{x}{x+1}$.

(b) $f : \mathbb{R}^+ \to \mathbb{R}$, where $f(x) = \dfrac{1}{x}$.

Solution:

(a)

$$f(x) = 1 - \frac{1}{x+1}$$

For any x, y with $x < y$ and $x, y \in [0, +\infty)$, we have $f(x) < f(y)$. Then $f(x)$ is $\boxed{\text{strictly increasing}}$.
For $x \in [0, +\infty)$, $0 = f(0) \leq f(x) \leq \lim_{x \to +\infty} f(x) = 1$. Then $f(x)$ is $\boxed{\text{bounded}}$.

(b) For any x, y with $x < y$ and $x, y \in (0, +\infty)$, we have $f(x) > f(y)$. Therefore, f is $\boxed{\text{strictly decreasing}}$.
Clearly, $f(x) = 1/x > 0$ for any $x \in \mathbb{R}^+$. So f is $\boxed{\text{bounded below}}$ by 0. On the other hand, f is not bounded above. Otherwise, if $f(x) \leq M$ for any $x \in \mathbb{R}^+$, then, in particular, $M + 1 = f(1/(M+1)) \leq M$, which is a contradiction.

10. Find whether the function is even, odd or neither:

(a) (Optional) $f(x) = x^2 - |x|$

(b) $f(x) = \log_2 \left(x + \sqrt{x^2 + 1}\right)$

(c) (Optional) $f(x) = x\left(\dfrac{a^x - 1}{a^x + 1}\right)$

(d) $f(x) = \sin\, x + \cos\, x$

Solution:

(a)

$$f(-x) = x^2 - |x| = f(x)$$

Thus, $f(x)$ is $\boxed{\text{even}}$.

(b)

$$\begin{aligned} f(-x) &= \log_2\left(-x + \sqrt{x^2+1}\right) \\ &= \log_2\left((-x + \sqrt{x^2+1}) \cdot \frac{x + \sqrt{x^2+1}}{x + \sqrt{x^2+1}}\right) \\ &= \log_2\left(\frac{1}{x + \sqrt{x^2+1}}\right) \\ &= -f(x) \end{aligned}$$

Thus, $f(x)$ is $\boxed{\text{odd}}$.

(c)

$$\begin{aligned} f(-x) &= -x(\frac{a^{-x} - 1}{a^{-x} + 1}) \\ &= x(\frac{a^x - 1}{a^x + 1}) \\ &= f(x) \end{aligned}$$

Thus, $f(x)$ is $\boxed{\text{even}}$.

(d)

$$\begin{aligned} f(-x) &= \sin(-x) + \cos(-x) \\ &= -\sin x + \cos x \end{aligned}$$

$f(x)$ is $\boxed{\text{neither even nor odd}}$ since $f(-x) \neq f(x)$ and $f(-x) \neq -f(x)$.

11. Without using L'Hôpital's rule, evaluate the limit, if it exists. If not, determine whether the one-sided limits exist (finite or infinite).

(a) $\lim\limits_{x \to 3} \dfrac{x^3 - 3x^2 + 5x - 15}{x^2 - x - 12}$.

(b) (Optional) $\lim\limits_{x \to 1/2} \dfrac{1 - 32x^5}{1 - 8x^3}$.

(c) (Optional) $\lim_{x \to 1} \dfrac{x - \sqrt{2 - x^2}}{2x - \sqrt{2 + 2x^2}}$.

(d) $\lim_{x \to 1} \dfrac{\sqrt{x^2 + 8} - \sqrt{10 - x^2}}{\sqrt{x^2 + 3} - \sqrt{5 - x^2}}$.

(e) (Optional) $\lim_{x \to 1} \left(\dfrac{2}{1 - x^2} + \dfrac{1}{x - 1} \right)$.

(f) $\lim_{x \to a} \left(\dfrac{2a}{x^2 - a^2} - \dfrac{1}{x - a} \right)$.

(g) $\lim_{x \to a} \left(\dfrac{x^m - a^m}{x^n - a^n} \right)$.

(h) $\lim_{x \to 1} \left(\dfrac{x - 1}{x^{1/4} - 1} \right)$.

(i) (Optional) $\lim_{x \to 0} \left(\dfrac{x^{7/10} + 3x^{4/3} + 2x}{x^{1/3} + 4x^{2/3} + 2x^{1/5}} \right)$.

Solution:

(a)

$$\begin{aligned}
&\lim_{x \to 3} \frac{x^3 - 3x^2 + 5x - 15}{x^2 - x - 12} \\
&= \frac{3^3 - 3(3^2) + 5(3) - 15}{3^2 - 3 - 12} \\
&= \boxed{0}
\end{aligned}$$

(b)

$$\begin{aligned}
&\lim_{x \to 1/2} \frac{1 - 32x^5}{1 - 8x^3} \\
&= \lim_{x \to 1/2} \frac{(1 - 2x)(1 + 2x + 4x^2 + 8x^3 + 16x^4)}{(1 - 2x)(1 + 2x + 4x^2)} \\
&= \lim_{x \to 1/2} \frac{1 + 2x + 4x^2 + 8x^3 + 16x^4}{1 + 2x + 4x^2} \\
&= \frac{1 + 2(\frac{1}{2}) + 4(\frac{1}{2})^2 + 8(\frac{1}{2})^3 + 16(\frac{1}{2})^4}{1 + 2(\frac{1}{2}) + 4(\frac{1}{2})^2} \\
&= \boxed{\frac{5}{3}}
\end{aligned}$$

(c)

$$\begin{aligned}
&\lim_{x\to1}\frac{x-\sqrt{2-x^2}}{2x-\sqrt{2+2x^2}}\\
&=\lim_{x\to1}\frac{x-\sqrt{2-x^2}}{2x-\sqrt{2+2x^2}}\cdot\frac{x+\sqrt{2-x^2}}{2x+\sqrt{2+2x^2}}\cdot\frac{2x+\sqrt{2+2x^2}}{x+\sqrt{2-x^2}}\\
&=\lim_{x\to1}\frac{x^2-(2-x^2)}{4x^2-(2+2x^2)}\cdot\frac{2x+\sqrt{2+2x^2}}{x+\sqrt{2-x^2}}\\
&=\lim_{x\to1}\frac{2x^2-2}{2x^2-2}\cdot\frac{2x+\sqrt{2+2x^2}}{x+\sqrt{2-x^2}}\\
&=\lim_{x\to1}\frac{2x+\sqrt{2+2x^2}}{x+\sqrt{2-x^2}}\\
&=\frac{2(1)+\sqrt{2+2(1)^2}}{1+\sqrt{2-1^2}}\\
&=\boxed{2}
\end{aligned}$$

(d)

$$\begin{aligned}
&\lim_{x\to1}\frac{\sqrt{x^2+8}-\sqrt{10-x^2}}{\sqrt{x^2+3}-\sqrt{5-x^2}}\\
&=\lim_{x\to1}\frac{\sqrt{x^2+8}-\sqrt{10-x^2}}{\sqrt{x^2+3}-\sqrt{5-x^2}}\cdot\frac{\sqrt{x^2+8}+\sqrt{10-x^2}}{\sqrt{x^2+3}+\sqrt{5-x^2}}\cdot\frac{\sqrt{x^2+3}+\sqrt{5-x^2}}{\sqrt{x^2+8}+\sqrt{10-x^2}}\\
&=\lim_{x\to1}\frac{x^2+8-(10-x^2)}{x^2+3-(5-x^2)}\cdot\frac{\sqrt{x^2+3}+\sqrt{5-x^2}}{\sqrt{x^2+8}+\sqrt{10-x^2}}\\
&=\lim_{x\to1}\frac{\sqrt{x^2+3}+\sqrt{5-x^2}}{\sqrt{x^2+8}+\sqrt{10-x^2}}\\
&=\frac{\sqrt{1^2+3}+\sqrt{5-1^2}}{\sqrt{1^2+8}+\sqrt{10-1^2}}\\
&=\boxed{\frac{2}{3}}
\end{aligned}$$

(e)

$$\begin{aligned}
&\lim_{x\to1}\left(\frac{2}{1-x^2}+\frac{1}{x-1}\right)\\
&=\lim_{x\to1}\frac{2-(1+x)}{(1-x)(1+x)}\\
&=\lim_{x\to1}\frac{1}{1+x}\\
&=\frac{1}{1+1}\\
&=\boxed{\frac{1}{2}}
\end{aligned}$$

(f)

$$\begin{aligned}&\lim_{x\to a}\left(\frac{2a}{x^2-a^2}-\frac{1}{x-a}\right)\\&=\lim_{x\to a}\frac{2a-(x+a)}{(x-a)(x+a)}\\&=\lim_{x\to a}\frac{-1}{x+a}\end{aligned}$$

(Case 1) If $a \neq 0$,

$$\lim_{x\to a}\frac{-1}{x+a}=\frac{-1}{a+a}=\boxed{-\frac{1}{2a}}$$

(Case 2) If $a = 0$, the limit $\boxed{\text{does not exist}}$ since

$$\lim_{x\to a^-}\frac{-1}{x+a}=\lim_{x\to 0^-}\frac{-1}{x}=+\infty$$

while

$$\lim_{x\to a^+}\frac{-1}{x+a}=\lim_{x\to 0^+}\frac{-1}{x}=-\infty$$

(g) (Case 1) Suppose $a \neq 0$.

- If $n \neq 0$:
  - If $m = 0$, then
$$\frac{x^m-a^m}{x-a}=\frac{1-1}{x-a}=0.$$
  - If $m > 0$, then
$$\lim_{x\to a}\frac{x^m-a^m}{x-a}=\lim_{x\to a}\sum_{k=0}^{m-1}x^k a^{m-1-k}=\sum_{k=0}^{m-1}a^{m-1}=ma^{m-1}.$$
  - If $m < 0$, then by the above limit,
$$\lim_{x\to a}\frac{x^m-a^m}{x-a}=\lim_{x\to a}-x^m a^m\cdot\frac{x^{-m}-a^{-m}}{x-a}=-a^{2m}(-m)a^{-m-1}=ma^{m-1}.$$

  Hence, if $n \neq 0$, we have
$$\lim_{x\to a}\frac{x^m-a^m}{x^n-a^n}=\lim_{x\to a}\frac{x^m-a^m}{x-a}\cdot\frac{x-a}{x^n-a^n}=\boxed{\frac{m}{n}a^{m-n}}.$$
- If $n = 0$, $\dfrac{x^m-a^m}{x^n-a^n}=\dfrac{x^m-a^m}{0}$ is not defined and so the limit $\boxed{\text{does not exist}}$.

(Case 2) Suppose $a = 0$.

- If $m = n$:
$$\lim_{x\to a}\frac{x^m-a^m}{x^n-a^n}=\boxed{1}$$

- If $m > n$:

$$\lim_{x\to a} \frac{x^m - a^m}{x^n - a^n} = \lim_{x\to 0} x^{m-n} = \boxed{0}$$

- If $m < n$: The limit $\boxed{\text{does not exist}}$ since

$$\lim_{x\to a^+} \frac{x^m - a^m}{x^n - a^n} = \lim_{x\to 0^+} \frac{1}{x^{n-m}} = +\infty,$$

while

$$\lim_{x\to a^-} \frac{x^m - a^m}{x^n - a^n} = \lim_{x\to 0^-} \frac{1}{x^{n-m}} = -\infty.$$

(h)

$$\begin{aligned} &\lim_{x\to 1} \frac{x-1}{x^{1/4}-1} \\ =&\lim_{x\to 1} \frac{(x^{1/4}-1)(x^{1/4}+1)(x^{1/2}+1)}{x^{1/4}-1} \\ =&\lim_{x\to 1}(x^{1/4}+1)(x^{1/2}+1) \\ =&(1+1)(1+1) \\ =&\boxed{4} \end{aligned}$$

(i)

$$\begin{aligned} &\lim_{x\to 0} \frac{x^{7/10}+3x^{4/3}+2x}{x^{1/3}+4x^{2/3}+2x^{1/5}} \\ =&\lim_{x\to 0} \frac{x^{1/2}+3x^{17/15}+2x^{4/5}}{x^{2/15}+4x^{7/15}+2} \\ =&\frac{0+0+0}{0+0+2} \\ =&\boxed{0} \end{aligned}$$

12. Without using L'Hôpital's rule, evaluate the limit, if it exists. If not, determine whether the one-sided limits exist (finite or infinite).

(a) $\displaystyle\lim_{x\to\infty} \frac{\sqrt{x^4+1}-\sqrt{x^4-1}}{x}$.

(b) $\displaystyle\lim_{x\to\infty} \frac{\sqrt{3x^2-1}-\sqrt{2x^2+1}}{4x+3}$.

(c) $\displaystyle\lim_{x\to\pi/2} \left(\frac{1-\sin^3 x}{1-\sin^2 x}\right)$.

(d) $\displaystyle\lim_{x\to\pi/4} \left(\frac{\sin 2x-(1+\cos(2x))}{\cos x-\sin x}\right)$.

(e) $\lim_{x\to\pi/4} \dfrac{\sqrt{2}-\cos x-\sin x}{(4x-\pi)^2}$.

(f) $\lim_{x\to 0} \dfrac{\sin 7x-\sin x}{\sin 6x}$.

(g) $\lim_{x\to 0} \left(\dfrac{1+x}{1-x}\right)^{1/x}$.

(h) $\lim_{x\to 0} \left(\dfrac{\sqrt{x+1}-1}{\ln(1+x)}\right)$.

(i) $\lim_{x\to 0} \left(\dfrac{e^{ax}-e^{a}}{x}\right)$ where a is a constant.

Solution:

(a)

$$\begin{aligned}\lim_{x\to\infty} \frac{\sqrt{x^4+1}-\sqrt{x^4-1}}{x} &= \lim_{x\to\infty} \frac{(\sqrt{x^4+1}-\sqrt{x^4-1})(\sqrt{x^4+1}+\sqrt{x^4-1})}{x(\sqrt{x^4+1}+\sqrt{x^4-1})}\\ &= \lim_{x\to\infty} \frac{2}{x(\sqrt{x^4+1}+\sqrt{x^4-1})}\\ &= \boxed{0}\end{aligned}$$

(b)

$$\begin{aligned}\lim_{x\to\infty} \frac{\sqrt{3x^2-1}-\sqrt{2x^2+1}}{4x+3} &= \lim_{x\to\infty} \frac{\sqrt{3-\dfrac{1}{x^2}}-\sqrt{2+\dfrac{1}{x^2}}}{4+\dfrac{3}{x}}\\ &= \boxed{\frac{\sqrt{3}-\sqrt{2}}{4}}\end{aligned}$$

(c)

$$x^3-1=(x-1)(x^2+x+1)$$

$$\begin{aligned}\lim_{x\to\pi/2} \left(\frac{1-\sin^3 x}{1-\sin^2 x}\right) &= \lim_{x\to\pi/2} \frac{(1-\sin x)(1+\sin x+\sin^2 x)}{(1-\sin x)(1+\sin x)}\\ &= \lim_{x\to\pi/2} \frac{(1+\sin x+\sin^2 x)}{(1+\sin x)}\\ &= \frac{1+1+1}{1+1}\\ &= \boxed{\frac{3}{2}}\end{aligned}$$

(d) Note that $1+\cos 2x = 1 + (2\cos^2 x - 1) = 2\cos^2 x$ and $\sin 2x = 2\sin x \cos x$. We have

$$\begin{aligned}
\lim_{x\to\pi/4}\left(\frac{\sin\,2x-(1+\cos\,2x)}{\cos\,x-\sin\,x}\right) &= \lim_{x\to\pi/4}\frac{2\cos x(\sin x-\cos x)}{\cos x-\sin x}\\
&= \lim_{x\to\pi/4} -2\cos x\\
&= \boxed{-\sqrt{2}}
\end{aligned}$$

(e) Let $y = 4x - \pi$, then we have $x = \dfrac{y+\pi}{4}$. Also, note that $x \to \dfrac{\pi}{4} \iff y \to 0$.

Therefore, we have

$$\begin{aligned}
\lim_{x\to\pi/4}\frac{\sqrt{2}-\cos\,x-\sin\,x}{(4x-\pi)^2} &= \lim_{y\to 0}\frac{\sqrt{2}-\cos\frac{y+\pi}{4}-\sin\frac{y+\pi}{4}}{y^2}\\
&= \lim_{y\to 0}\frac{\sqrt{2}-\left(\cos\frac{y}{4}\cos\frac{\pi}{4}-\sin\frac{y}{4}\sin\frac{\pi}{4}\right)-\left(\sin\frac{y}{4}\cos\frac{\pi}{4}+\cos\frac{y}{4}\sin\frac{\pi}{4}\right)}{y^2}\\
&= \lim_{y\to 0}\frac{\sqrt{2}-\left(\frac{1}{\sqrt{2}}\cos\frac{y}{4}-\frac{1}{\sqrt{2}}\sin\frac{y}{4}\right)-\left(\frac{1}{\sqrt{2}}\sin\frac{y}{4}+\frac{1}{\sqrt{2}}\cos\frac{y}{4}\right)}{y^2}\\
&= \lim_{y\to 0}\frac{\sqrt{2}-\frac{2}{\sqrt{2}}\cos\frac{y}{4}}{y^2}\\
&= \sqrt{2}\left(\lim_{y\to 0}\frac{1-\cos\frac{y}{4}}{y^2}\right)\\
&= \sqrt{2}\left(\lim_{y\to 0}\frac{2\sin^2\frac{y}{8}}{y^2}\right)\\
&= 2\sqrt{2}\left(\lim_{y\to 0}\frac{\sin^2\frac{y}{8}}{\left(\frac{y}{8}\right)^2\cdot 8^2}\right)\\
&= \frac{2\sqrt{2}}{64}\cdot 1^2 \quad \left(\text{since } \lim_{y\to 0}\frac{\sin y}{y}=1\right)\\
&= \boxed{\frac{\sqrt{2}}{32}}
\end{aligned}$$

(f)

$$\begin{aligned}
\lim_{x\to 0}\frac{\sin\,7x-\sin\,x}{\sin\,6x} &= \lim_{x\to 0}\frac{\frac{1}{x}(\sin\,7x-\sin\,x)}{\frac{1}{x}(\sin\,6x)}\\
&= \lim_{x\to 0}\frac{\frac{\sin 7x}{7x}\cdot 7-\frac{\sin x}{x}}{\frac{\sin 6x}{6x}\cdot 6}\\
&= \frac{1\cdot 7-1}{1\cdot 6} \quad \left(\text{since } \lim_{x\to 0}\frac{\sin x}{x}=1\right)\\
&= \frac{6}{6} = \boxed{1}
\end{aligned}$$

(g)

$$\begin{aligned}\lim_{x\to 0}\left(\frac{1+x}{1-x}\right)^{1/x} &= \lim_{x\to 0}(1+x)^{1/x}(1-x)^{1/(-x)}\\ &= \lim_{x\to 0}\left(1+\frac{1}{\frac{1}{x}}\right)^{1/x}\left(1+\frac{1}{\frac{1}{-x}}\right)^{1/(-x)}\\ &= e\cdot e \quad \left(\text{since } \lim_{y\to\infty}\left(1+\frac{1}{y}\right)^{y}=e\right)\\ &= \boxed{e^2}\end{aligned}$$

(h)

$$\begin{aligned}\lim_{x\to 0}\left(\frac{\sqrt{x+1}-1}{\ln(1+x)}\right) &= \lim_{x\to 0}\frac{x}{\ln(x+1)}\cdot\frac{\sqrt{x+1}-1}{x}\\ &= \lim_{x\to 0}\frac{x}{\ln(x+1)}\cdot\frac{(\sqrt{x+1}-1)(\sqrt{x+1}+1)}{x(\sqrt{x+1}+1))}\\ &= \lim_{x\to 0}\frac{x}{\ln(x+1)}\cdot\frac{1}{\sqrt{x+1}+1}\\ &= 1\cdot\frac{1}{\sqrt{0+1}+1}\\ &= \boxed{\frac{1}{2}}\end{aligned}$$

(i) (Case 1) Suppose $a=0$. We have

$$\lim_{x\to 0}\left(\frac{e^{ax}-e^{a}}{x}\right)=\lim_{x\to 0}\left(\frac{1-1}{x}\right)=\lim_{x\to 0}\left(\frac{0}{x}\right)=\boxed{0}.$$

(Case 2) Suppose $a\neq 0$. We have

$$\begin{aligned}\lim_{x\to 0}\left(\frac{e^{ax}-e^{a}}{x}\right) &= \lim_{x\to 0}\frac{e^{ax}-1+1-e^{a}}{x}\\ &= \lim_{x\to 0}\left(\left(a\frac{e^{ax}-1}{ax}\right)+\frac{1-e^{a}}{x}\right)\end{aligned}$$

Now, $\lim_{x\to 0}\frac{e^{ax}-1}{ax}=1$ while $\lim_{x\to 0}\frac{1}{x}=\infty$. Also, note that $1-e^{a}\neq 0$ as $a\neq 0$.

We conclude that the limit $\lim_{x\to 0}\left(\frac{e^{ax}-e^{a}}{x}\right)$ $\boxed{\text{does not exist}}$.

We now consider the one-sided limits. We have

$$\lim_{x\to 0^+}\frac{1-e^{a}}{x}=\begin{cases}+\infty & \text{if } a<0\\ -\infty & \text{if } a>0\end{cases} \quad\text{and}\quad \lim_{x\to 0^-}\frac{1-e^{a}}{x}=\begin{cases}-\infty & \text{if } a<0\\ +\infty & \text{if } a>0\end{cases}$$

and hence

$$\lim_{x\to 0^+}\left(\frac{e^{ax}-e^{a}}{x}\right)=\boxed{\begin{cases}+\infty & \text{if } a<0\\ -\infty & \text{if } a>0\end{cases}} \quad\text{and}\quad \lim_{x\to 0^-}\left(\frac{e^{ax}-e^{a}}{x}\right)=\boxed{\begin{cases}-\infty & \text{if } a<0\\ +\infty & \text{if } a>0\end{cases}}.$$

13. Evaluate the following limits.

(a) $\lim_{x\to 0^-} x|\sin\frac{1}{x}|$

(b) $\lim_{x\to +\infty} \dfrac{\sin(\tan x)+\tan(\sin x)}{x+1}$

Solution:

(a) Note that $0 \le \left|\sin\dfrac{1}{x}\right| \le 1$ and so $-x \le x\left|\sin\dfrac{1}{x}\right| \le x$.
Since $\lim_{x\to 0} -x = 0$ and $\lim_{x\to 0} x = 0$,
by squeeze theorem, $\lim_{x\to 0} x\left|\sin\dfrac{1}{x}\right| = 0$.
Therefore, $\lim_{x\to 0^-} x\left|\sin\dfrac{1}{x}\right| = \boxed{0}$.

(b) Note that $-1 \le \sin x \le 1$ for any x, and so

$$-1 \le \sin(\tan x) \le 1.$$

Also, as $\tan(x)$ is increasing in $[-1, 1]$, we have

$$\tan(-1) \le \tan(\sin x) \le \tan 1.$$

Therefore, we have

$$\frac{-1+\tan(-1)}{x+1} \le \frac{\sin(\tan x)+\tan(\sin x)}{x+1} \le \frac{1+\tan 1}{x+1} \text{ for } x > 0.$$

Since $\lim_{x\to +\infty} \dfrac{-1+\tan(-1)}{x+1} = 0$ and $\lim_{x\to +\infty} \dfrac{1+\tan 1}{x+1} = 0$,
by squeeze theorem, $\lim_{x\to +\infty} \dfrac{\sin(\tan x)+\tan(\sin x)}{x+1} = \boxed{0}$.

14. Evaluate the following limits.

(a) $\lim_{x\to 0} \dfrac{\tan x - \sin x}{\sin^3 x}$

(b) $\lim_{x\to 0} \dfrac{\tan^2 x}{\sin(x^2)}$

(c) $\lim_{x\to 0} \dfrac{1-\cos x\sqrt{\cos 2x}}{x^2}$

Solution:

(a)

$$\begin{aligned}
\lim_{x\to 0}\frac{\tan x-\sin x}{\sin^3 x} &= \lim_{x\to 0}\frac{\frac{\sin x}{\cos x}-\sin x}{\sin^3 x}\\
&= \lim_{x\to 0}\frac{\frac{1}{\cos x}-1}{\sin^2 x}\\
&= \lim_{x\to 0}\frac{1-\cos x}{\sin^2 x\cos x}\\
&= \lim_{x\to 0}\frac{1-\cos x}{(1-\cos^2 x)\cos x}\\
&= \lim_{x\to 0}\frac{1-\cos x}{(1-\cos x)(1+\cos x)\cos x}\\
&= \lim_{x\to 0}\frac{1}{(1+\cos x)\cos x}\\
&= \frac{1}{(1+1)(1)}\\
&= \boxed{\frac{1}{2}}
\end{aligned}$$

(b)

$$\begin{aligned}
\lim_{x\to 0}\frac{\tan^2 x}{\sin(x^2)} &= \lim_{x\to 0}\left(\frac{\frac{\tan^2 x}{x^2}}{\frac{\sin(x^2)}{x^2}}\right)\\
&= \lim_{x\to 0}\left(\frac{\frac{\sin x}{x}\cdot\frac{\sin x}{x}\cdot\frac{1}{\cos^2 x}}{\frac{\sin(x^2)}{x^2}}\right)\\
&= \frac{1\cdot 1\cdot\frac{1}{1}}{1}\\
&= \boxed{1}
\end{aligned}$$

(c)

$$\begin{aligned}
\lim_{x\to 0}\frac{1-\cos x\sqrt{\cos 2x}}{x^2} &= \lim_{x\to 0}\frac{1-\cos x\sqrt{\cos 2x}}{x^2}\cdot\frac{1+\cos x\sqrt{\cos 2x}}{1+\cos x\sqrt{\cos 2x}}\\
&= \lim_{x\to 0}\frac{1-\cos^2 x\cos 2x}{x^2(1+\cos x\sqrt{\cos 2x})}\\
&= \lim_{x\to 0}\frac{1-(1-\sin^2 x)(1-2\sin^2 x)}{x^2(1+\cos x\sqrt{\cos 2x})}\\
&= \lim_{x\to 0}\frac{1-(1-3\sin^2 x+2\sin^4 x)}{x^2(1+\cos x\sqrt{\cos 2x})}\\
&= \lim_{x\to 0}\frac{3\sin^2 x-2\sin^4 x}{x^2(1+\cos x\sqrt{\cos 2x})}\\
&= \lim_{x\to 0}\left[\left(\frac{\sin x}{x}\right)^2\cdot\frac{3-2\sin^2 x}{1+\cos x\sqrt{\cos 2x}}\right]\\
&= \left[\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2\right]\left[\lim_{x\to 0}\frac{3-2\sin^2 x}{1+\cos x\sqrt{\cos 2x}}\right]\\
&= (1)^2\cdot\frac{3-2\cdot 0}{1+1\cdot 1}\\
&= \boxed{\frac{3}{2}}
\end{aligned}$$