MATH 1010F Lecture 25

30/11/2023

Last time

- Fundamental theorem of calculus
  - 1^{st} FTC: $\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x)$
  - 2^{nd} FTC: $\int_a^b f(x)\,dx = F(b) - F(a)$
- Techniques:
  - substitution
  - integration by parts
  - $\frac{d}{dx}\left(\int_{u(x)}^{v(x)} f(t)\,dt\right) = f(v(x))\,v'(x) - f(u(x))\,u'(x)$
  - $\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right) = \int_0^1 f(x)\,dx$
  - integrating odd/even functions over $[-a, a]$

// Other symmetry arguments

Example $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin^{2023} x}{\sin^{2023} x + \cos^{2023} x}\,dx = ?$

Solution Let $u = \frac{\pi}{2} - x$, then $du = -dx$ and

$$\int_0^{\frac{\pi}{2}} \frac{\sin^{2023} x}{\sin^{2023} x + \cos^{2023} x}\,dx$$

$$= \int_{\frac{\pi}{2}}^{0} \frac{\sin^{2023}\left(\frac{\pi}{2} - u\right)}{\sin^{2023}\left(\frac{\pi}{2} - u\right) + \cos^{2023}\left(\frac{\pi}{2} - u\right)} \cdot (-1)\,du$$

$$= -\int_{\frac{\pi}{2}}^{0} \frac{\cos^{2023} u}{\cos^{2023} u + \sin^{2023} u} \, du \quad \left(\because \begin{array}{l} \sin\left(\frac{\pi}{2}-u\right) = \cos u \\ \cos\left(\frac{\pi}{2}-u\right) = \sin u \end{array} \right)$$

$$= \int_{0}^{\frac{\pi}{2}} \frac{\cos^{2023} u}{\cos^{2023} u + \sin^{2023} u} \, du$$

Now, we have

$$\int_{0}^{\frac{\pi}{2}} \frac{\sin^{2023} x}{\sin^{2023} x + \cos^{2023} x} + \int_{0}^{\frac{\pi}{2}} \frac{\cos^{2023} \underline{\underline{u}}}{\cos^{2023} \underline{\underline{u}} + \sin^{2023} \underline{\underline{u}}} \, d\underline{\underline{u}}$$

just a dummy variable, can be replaced with x

$$= \int_{0}^{\frac{\pi}{2}} \frac{\sin^{2023} x + \cos^{2023} x}{\sin^{2023} x + \cos^{2023} x} \, dx$$

$$= \int_{0}^{\frac{\pi}{2}} 1 \, dx$$

$$= \frac{\pi}{2}$$

$$\therefore \int_{0}^{\frac{\pi}{2}} \frac{\sin^{2023} x}{\sin^{2023} x + \cos^{2023} x} \, dx = \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi}{4} \quad //$$