Fuzzifying the Constraint Hierarchies Framework

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Abstract. The Constraint Hierarchy (CH) framework is used to tackle *multiple criteria selection* (MCS), consisting of a set of candidates and a set of, possibly competing, criteria for selecting the "best" candidate(s). In this paper, we identify aspects of the CH framework for further enhancement so as to model and solve MCS problems more accurately. We propose the Fuzzy Constraint Hierarchies framework, which allows constraints to belong to, possibly, more than one level in a constraint hierarchy to a varying degree. We also propose to replace the standard equality relation $=$ used in valuation comparators of the CH framework by the α-approximate equality relation $=_{a(\alpha)}$ for providing more flexible control over the handling of valuations with close error values. These proposals result in three new classes of valuation comparators. Formal properties of the new comparators are given, wherever possible.

1 Introduction

An *over-constrained system* [3] is a set of constraints with no solution, caused by some of the constraints contradicting others. From the users' point of view, no-solution is hardly a satisfactory answer. Users normally want solutions of some sort, even if they are *partial* in the sense that the solutions can cause some constraints to be violated. The task becomes how to relax or weaken some of the constraints in an over-constrained system so as to obtain useful partial solution.

New solution techniques, well summarized in [3], are developed to model and tackle over-constrained systems. In particular, the *Constraint Hierarchies* (CH) framework by Borning *et al.* [2] allows constraints to be *required* and *preferential*. While the required constraints must be satisfied, satisfaction of preferential constraints are not mandatory but they should be satisfied as much as possible. Each preferential constraint has a strength and, together with the required constraints, they form a *constraint hierarchy*. A *better* comparator is used to rank and select among competing possible solutions.

The problem at hand is that of *multiple criteria selection* (MCS), which consists of a set of candidates and a set of, possibly competing, criteria for selecting the "best" candidate(s). MCS is a common decision process in answering such questions as which presidency candidate to vote for, which applicants to admit to the graduate programme, and so on. The CH framework is chosen to model and solve MCS problems since selection criteria are like preferential constraints:

(1) criteria have relative priorities and (2) candidates are most likely not able to satisfy all criteria anyway and we would like to select the candidate that satisfy the most constraints in some sense. There are two issues in tackling MCS problems by the CH framework. First, it is sometimes difficult to determine the relative strength of some criteria with absolute certainty. Second, the standard equality relation used in *better* comparators is sometimes too rigid and presents problems in handling competing solutions with very close error values.

In this paper, we show how simple application of fuzzy techniques can resolve the issues. To help better model MCS problems, we propose to fuzzify the levels of constraint hierarchies, allowing constraints to belong to a level with a certain degree of membership. To deal with valuations with close error values, we propose to use the α-approximate equality relation $=_{a(\alpha)}$, a relaxed form of equality, in the comparators, obtained by fuzzifying and de-fuzzifying the standard equality relation $=$. The result is the *Fuzzy Constraint Hierarchy* framework plus three new classes of valuation comparators, each of which improves upon their counterparts in the CH framework.

The paper is organized as follows. We first give a brief overview of the CH framework and two motivating scenarios for our work, followed by an exposition of the Fuzzy Constraint Hierarchies framework, its related class of comparators, and formal properties of these comparators. We also show how another class of comparators is defined using the α-approximate equality relation. A seamless integration of the two new classes of comparators gives the best of both worlds. Last but not least, we conclude the paper by listing our contributions and discussing future work and implementation issues.

2 The Constraint Hierarchies Framework

In this section, we give a brief introduction to the CH framework. Interested readers are referred to [2] for full details. Two example scenarios are used to illustrate aspects of the framework opted for further enhancement.

2.1 Constraint Hierarchies

Let $\mathcal{D}$ be the domain. An *n-ary constraint* c is a relation over $\mathcal{D}^n$. A *labeled constraint* is a pair (c, k), where c is a constraint and k is a label stating the *strength* of c. The strength labels are totally ordered. Without loss of generality, we map the strengths to non-negative integers. A *constraint hierarchy* H is a multiset of labeled constraints with a finite number of labels. According to the associated labels, constraints are classified into different disjoint multisets H_k, called *levels*, for each label k: level H_k contains the set of labeled constraints with label k. Given a constraint hierarchy, we use H_0 to denote the set of *required* (or *hard*) constraints. The levels $H_1, H_2, \ldots, H_n$ denote different sets of *preferential* (or *soft*) constraints, ranging from the strongest strength to the weakest.

A *valuation* for a set of constraints is a function that maps the free variables in the constraints to elements in the domain over which the constraints are defined.

Let c be a constraint and θ a valuation. The expression $c\theta$ is the boolean result of applying θ to c. We say that $c\theta$ *holds* if the value of $c\theta$ is **true**.

We define $S_0 = \{\theta \,|\, \forall c \in H_0 \text{ s.t. } c\theta \text{ holds}\}$ to be the set of valuations that satisfy the required constraints. The solution set to a constraint hierarchy H depends on the definition of a valuation comparator $better(\theta, \sigma, H)$ that compares two valuations θ and σ for the constraints in H. Many definitions of *better* are possible as long as the comparator is irreflexive, transitive, and respect the hierarchy [2] in the loose sense that a "better" valuation must indeed satisfy "more" constraints in H. Note that *better* does not define a total order since some valuations can be *incomparable* according to this definition. In other words, it is possible that a valuation θ is *not better* than valuation σ and vice versa. The *solution set* of a constraint hierarchy H is obtained by eliminating all potential valuations that are worse than some other potential valuations: $S = \{\theta \,|\, \theta \in S_0 \wedge \forall \sigma \in S_0 \text{ s.t. } \neg better(\sigma, \theta, H)\}$.

In the following, we give several definitions of *better*, which require an *error function* $e(c\theta)$ to measure how well a constraint c is satisfied by valuation θ. The error function returns non-negative real numbers and must satisfy the property: $e(c\theta) = 0 \Leftrightarrow c\theta$ holds. Given a constraint Hierarchy H and two valuations θ and σ, the comparator *locally-better* *l-b* is defined as follows:

$$\begin{aligned}
&l\text{-}b(\theta, \sigma, H) \equiv \exists k > 0 \text{ such that} \\
&\quad \forall i \in \{1, \ldots, k-1\} \forall p \in H_i \; e(p\theta) = e(p\sigma) \\
&\wedge \exists q \in H_k \; e(q\theta) < e(q\sigma) \\
&\wedge \forall r \in H_k \; e(r\theta) \leq e(r\sigma)
\end{aligned}$$

The *globally-better* schema *g-b* defines a class of global comparators, parametrized by a function g that combines the errors of all constraints in a level H_i.

$$\begin{aligned}
&g\text{-}b(\theta, \sigma, H, g) \equiv \exists k > 0 \text{ such that} \\
&\quad \forall i \in \{1, \ldots, k-1\} \; g(\theta, H_i) = g(\sigma, H_i) \\
&\wedge g(\theta, H_k) < g(\sigma, H_k)
\end{aligned}$$

Suppose we assign also a positive real number w_p to each constraint p to denote the *weight* of p. Three global comparators, namely *weighted-sum-better* *w-s-b*, *worst-case-better* *w-c-b*, and *least-square-better* *l-s-b* can be defined using different combining functions g.

$$\begin{aligned}
w\text{-}s\text{-}b(\theta, \sigma, H) &\equiv g\text{-}b(\theta, \sigma, H, g) \\
\text{where } g(\tau, H_i) &\equiv \sum_{p \in H_i} w_p e(p\tau)
\end{aligned}
\qquad
\begin{aligned}
l\text{-}s\text{-}b(\theta, \sigma, H) &\equiv g\text{-}b(\theta, \sigma, H, g) \\
\text{where } g(\tau, H_i) &\equiv \sum_{p \in H_i} w_p e(p\tau)^2
\end{aligned}$$

$$\begin{aligned}
w\text{-}c\text{-}b(\theta, \sigma, H) &\equiv g\text{-}b(\theta, \sigma, H, g) \\
\text{where } g(\tau, H_i) &\equiv \max\{w_p e(p\tau) \,|\, p \in H_i\}
\end{aligned}$$

Different specific comparators can be defined by choosing the appropriate error function e. We call this class of comparators *classical comparators* to differentiate them from the ones introduced later in this paper,

2.2 Dominance of Strong Constraints

What millions died—that Caesar might be great!
Thomas Campbell (1777–1844)

In the CH framework, two valuations are compared level by level, starting from the strongest level. If two valuations are equivalent in one level, then comparison proceeds to the next level. The comparison process stops once a valuation is found to be better than the other at a certain level, or when none of the two valuations are better than the other. Constraints in weaker levels are completely ignored. There are two problems. First, it is sometimes difficult to determine the relative strengths among constraints with absolute certainty in practice. Second, a constraint c_1 being more important than another constraint c_2 may not imply that the satisfaction of c_1 should take *strict* precedence over that of c_2. Failure to cope with these phenomena may result in counter-intuitive solutions.

Example 1 Suppose a company wants to promote a sales representative to become the general sales manager from a pool of candidates. The ideal candidate must be a degree holder. (S)he should have at least 5 years of working experience and be able to meet a sales quota of 1 million per annum. Being in a supervisory role, the potential manager should also be familiar with and be able to sell at least 20 products of the company. While the education background is a firm requirement, working experience and sales performance are, in general, considered to be more important than versatility in product range. In considering candidates with similar working experience and sales performance, the last criterion should also be taken into account, although to a lesser degree, to differentiate the best candidate from the rest.

The last criterion is common sense but is difficult to model in the CH framework. We model the problem to its nearest approximation as follows. There are four variables: D to denote if the candidate is a degree holder, Y to denote the candidate's working experience in number of years, Q to denote the candidate's sales figure in thousands of dollars, and P to denote the number of products that the candidate can sell. We get the following constraint hierarchy H.

Level	Constraints
H_0	$(c_1)\ D = \texttt{degree}$
H_1	$(c_2)\ Y \geq 5$, $(c_3)\ Q \geq 1000$
H_2	$(c_4)\ P \geq 20$

The following error function e measures how well the constraints are satisfied:

$$e(c_1\theta) = \begin{cases} 1 \text{ if } D\theta = \texttt{degree} \\ 0 \text{ Otherwise} \end{cases} \qquad e(c_2\theta) = \begin{cases} \dfrac{5 - Y\theta}{5} & \text{if } Y\theta < 5 \\ 0 & \text{if } Y\theta \geq 5 \end{cases}$$

$$e(c_3\theta) = \begin{cases} \dfrac{1000 - Q\theta}{1000} & \text{if } Q\theta < 1000 \\ 0 & \text{if } Q\theta \geq 1000 \end{cases} \qquad e(c_4\theta) = \begin{cases} \dfrac{20 - P\theta}{20} & \text{if } P\theta < 20 \\ 0 & \text{if } P\theta \geq 20 \end{cases}$$

Consider two competing candidates A and B with their corresponding qualifications encoded in the valuations θ_A and θ_B respectively:

$$\theta_A = \{D \mapsto \texttt{degree}, Y \mapsto 5, Q \mapsto 1000, P \mapsto 2\}$$
$$\theta_B = \{D \mapsto \texttt{degree}, Y \mapsto 5, Q \mapsto 990, P \mapsto 30\}$$

Applying the valuations to the constraints in H and using the error function e just defined, we obtain the error sequences of the two candidates.

θ	$e(c_1\theta)$	$e(c_2\theta)$	$e(c_3\theta)$	$e(c_4\theta)$
θ_A	0	0	0	0.9
θ_B	0	0	0.01	0

Assuming the weight of all constraints to be 1.0, we can easily check that any classical comparators would conclude A to be the better candidate. According to the requirement of the selection criteria and common sense, however, candidate B should be selected since B can sell many more products than A while A's edge over B in sales performance is only marginal (ten thousand in one million is negligible in a big corporation!). We call this the *dominance* problem. In this example, the slight minimization of the error of one stronger constraint precedes the consideration of weaker constraints even if the resulting valuation will cause very large error in the weaker constraint.

The dominance problem cannot be solved by simply adjusting the weight since the error of c_2 is 0 in both cases. One might argue changing the error function to

$$e(c_3\theta) = \begin{cases} \dfrac{1000 - Q\theta}{1000} & \text{if } Q\theta < 1000 - 100 \\ 0 & \text{if } Q\theta \geq 1000 - 100 \end{cases}$$

or some similar function to increase the "tolerance" of the constraint c_3. Candidates A and B are now equal up to level 1 and their comparison has to be resolved using c_4 in level 2. This seems to solve the dominance problem for the current scenario.

Example 2 What if candidate B can sell only 3 products instead?

$$\theta_A = \{D \mapsto \texttt{degree}, Y \mapsto 5, Q \mapsto 1000, P \mapsto 2\}$$
$$\theta_B = \{D \mapsto \texttt{degree}, Y \mapsto 5, Q \mapsto 990, P \mapsto 3\}$$

With the new error function, candidate B will always be selected using *all* comparators defined. Depending on how errors are compared, however, candidate A, who is slightly better in sales performance and slightly worse in the number of products, should be as competitive as, if not better than, candidate B. The new error function blindly concludes that the error of a valuation for c_3 is 0 if the error is less a certain value. There are two problems. First, this method fails if both candidates have sizable but close error values. Second, the errors of c_4 for the valuations are not taken into account. This violates the last requirement of the promotion criteria.

We cannot massage the relative strengths of the constraints either since sales performance is in general more important than number of products as stated in the requirement. We propose to solve the dominance problem by allowing constraints to belong to different levels with varying membership grade so that weak constraints can have influence in stronger levels.

2.3 Stiff Equality in Better Comparators

Equality is in the eyes of the beholder.

In comparing valuations in the CH framework, comparison proceeds from one level to the other in the hierarchy only when the valuations are concluded to be equivalent in the current level. In a local comparator, two valuations are *equivalent* in a level if the errors computed from the two valuations for every constraint in the level are equal. In a global comparator, two valuations are *equivalent* in a level if the errors computed from the two valuations for all constraints in the level using the combining function g are equal. Both notions of equivalence use the standard equality relation $=$, which governs that any individual x can only be equal to itself and nothing else. In daily experience, however, we are sometimes satisfied with things that are "approximately equal to" each other. In many real-life situations, we cannot afford to be too rigid since physical data always contain errors so that absolute equality can never be attained.

Given two valuations. There could be cases in which the errors of two valuations for a level are very close to each other and yet one valuation is concluded to be better than the other or incomparable to the other. The errors are, however, so close that the conclusion is drawn only in the numerical, but not common, sense. There could well be more useful information in the next level to help make a better decision. We call this the *stiff equality problem*.

Example 3 Let us use the sales representative promotion problem for illustration again. We assume the same modeling of the problem as in examples 1 and 2. In our first scenario, the competing candidates have the following valuations:

$$\theta_A = \{D \mapsto \texttt{degree}, Y \mapsto 2.9, Q \mapsto 700, P \mapsto 30\}$$
$$\theta_B = \{D \mapsto \texttt{degree}, Y \mapsto 3, Q \mapsto 690, P \mapsto 2\}$$

Applying the valuations to the constraints in H gives the following error sequences.

θ	$e(c_1\theta)$	$e(c_2\theta)$	$e(c_3\theta)$	$e(c_4\theta)$
θ_A	0	0.42	0.3	0
θ_B	0	0.4	0.31	0.9

The *l-b* comparator would find the candidates to be incomparable. Assuming all constraint weights to be 1.0, the global comparators would conclude B to be the better candidate.

Example 4 Worst yet, in our second scenario, the candidates' profiles become:

$$\theta_A = \{D \mapsto \texttt{degree}, Y \mapsto 2.9, Q \mapsto 690, P \mapsto 30\}$$
$$\theta_B = \{D \mapsto \texttt{degree}, Y \mapsto 3, Q \mapsto 700, P \mapsto 2\}$$

θ	$e(c_1\theta)$	$e(c_2\theta)$	$e(c_3\theta)$	$e(c_4\theta)$
θ_A	0	0.42	0.31	0
θ_B	0	0.4	0.3	0.9

All comparators will conclude B to be the better candidate.

Common sense would suggest the contrary for both examples since the two candidates are close to indistinguishable in level H_1. The abysmal differences in the error values are inconclusive. The candidates should better be differentiated using information in level H_2. An approach to solve the stiff equality problem is to relax the equality relationship in the comparators so that two error values of similar magnitude are considered equal.

3 Fuzzifying Hierarchy Levels

The closer one looks, the fuzzier things are.

Our first extension aims to solve the dominance problem. We propose to generalize the CH framework by allowing a constraint to belong to possibly more than one level in the hierarchy to a varying degree. In the following, we briefly introduce some basic machineries of the fuzzy set theory [4], followed by the definition of the *Fuzzy Constraint Hierarchies* (FCH) framework and its formal properties.

3.1 Basic Fuzzy Set Theory

Every *crisp set* A is associated with a *characteristic function* $\mu_A : \mathcal{U} \to \{0, 1\}$, which determines whether an individual from the universal set $\mathcal{U}$ is a member or non-member of A:

$$\mu_A(x) = \begin{cases} 1 \text{ if and only if } x \in A \\ 0 \text{ if and only if } x \notin A \end{cases}$$

Fuzzy sets are usually obtained by generalizing the characteristic function so that $\mu_A : \mathcal{U} \to [0, 1]$. Given an element $x \in \mathcal{U}$, $\mu_A(x)$ denotes the degree of membership of x in the fuzzy set A. A larger value indicates a higher degree of membership. The generalized μ_A is called a *membership function*. Thus, each element of a fuzzy set A is a pair $(x, \mu_A(x))$. We write $(x, \mu_A(x)) \in A$ to mean that x belongs to A with *membership grade* $\mu_A(x)$. In practice, the membership function μ_A of a fuzzy set has to be estimated from partial information sampled from, possibly, incomplete data.

3.2 Fuzzy-Labeled Constraints

The FCH framework inherits most of the structure of the CH framework. The domain $\mathcal{D}$ and the notions of constraints c and strengths k remain unchanged. The strengths are totally ordered and we use non-negative integers as strengths. In the FCH framework, we generalize the *level* H_k of each strength k from a crisp set to a fuzzy set for $k \geq 1$. The required level remains a crisp set. A *fuzzy-labeled constraint* is a tuple $(c, k, \mu_k(c))$, where c is a constraint, $k \geq 1$ is the strength of c, and $\mu_k(c)$ is a real number in $[0, 1]$ indicating the membership grade of c

in H_k. The function μ_k is the membership function associated with the fuzzy set H_k. A *fuzzy constraint hierarchy* H consists of a multiset H_0 of required constraints and a multiset of fuzzy-labeled constraints with a finite number of labels. It is important to note that a constraint c can belong to more than one level with varying degree of membership.

The solution set S of a fuzzy constraint hierarchy is defined similarly: $S_0 = \{\theta \,|\, \forall c \in H_0 \text{ s.t. } c\theta \text{ holds}\}$ and $S = \{\theta \,|\, \theta \in S_0 \wedge \forall \sigma \in S_0 \text{ s.t. } \neg better(\sigma, \theta, H)\}$, except that the valuation comparator *better* must now take the membership grade of constraints into account. Assuming the same notion of error function $e(c\theta)$, we present the *fuzzy-locally-better* comparator *f-l-b* as follows:

$$\begin{aligned}
&f\text{-}l\text{-}b(\theta, \sigma, H) \equiv \exists k > 0 \text{ such that}\\
&\quad \forall i \in \{1, \ldots, k-1\} \forall (p, \mu_i(p)) \in H_i \;\; e(p\theta)\mu_i(p) = e(p\sigma)\mu_i(p)\\
&\quad \wedge \exists (q, \mu_k(q)) \in H_k \;\; e(q\theta)\mu_k(q) < e(q\sigma)\mu_k(q)\\
&\quad \wedge \forall (r, \mu_k(r)) \in H_k \;\; e(r\theta)\mu_k(r) \leq e(r\sigma)\mu_k(r)
\end{aligned}$$

The terms $\mu_i(p)$, $\mu_k(q)$, and $\mu_k(r)$ can be canceled out on both sides of the equality and inequalities. We leave the terms in the definition since they are essential for a further generalization of the comparator. Disregarding the terms for the moment, this definition is almost identical to that of the *l-b* comparator in the CH framework. There is a subtle difference between the two comparators: a constraint can appear in possibly more than one level in the hierarchy.

The *fuzzy-globally-better* schema *f-l-g-b* is similar to the classical version except that the combining function g must now incorporate the membership of the constraints in each level:

$$\begin{aligned}
&f\text{-}g\text{-}b(\theta, \sigma, H, g) \equiv \exists k > 0 \text{ such that}\\
&\quad \forall i \in \{1, \ldots, k-1\} \;\; g(\theta, H_i, \mu_i) = g(\sigma, H_i, \mu_i)\\
&\quad \wedge g(\theta, H_k, \mu_k) < g(\sigma, H_k, \mu_k)
\end{aligned}$$

The three classical global comparators can be fuzzified as follows:

$$\begin{aligned}
f\text{-}w\text{-}s\text{-}b(\theta, \sigma, H) &\equiv f\text{-}g\text{-}b(\theta, \sigma, H, g)\\
\text{where } g(\tau, H_i, \mu_i) &\equiv \sum_{(p, \mu_i(p)) \in H_i} w_p e(p\tau)\mu_i(p)
\end{aligned}$$

$$\begin{aligned}
f\text{-}w\text{-}c\text{-}b(\theta, \sigma, H) &\equiv f\text{-}g\text{-}b(\theta, \sigma, H, g)\\
\text{where } g(\tau, H_i, \mu_i) &\equiv \max\{w_p e(p\tau)\mu_i(p) \,|\, (p, \mu_i(p)) \in H_i\}
\end{aligned}$$

$$\begin{aligned}
f\text{-}l\text{-}s\text{-}b(\theta, \sigma, H) &\equiv f\text{-}g\text{-}b(\theta, \sigma, H, g)\\
\text{where } g(\tau, H_i, \mu_i) &\equiv \sum_{(p, \mu_i(p)) \in H_i} w_p e(p\tau)^2\mu_i(p)
\end{aligned}$$

We call the newly defined comparators *fuzzy comparators*.

3.3 Illustrative Examples

The sales representative promotion problem can be more accurately formulated in the FCH framework as follows:

Level	Constraints
H_0	$(c_1, 1.0)$
H_1	$(c_2, 1.0)$, $(c_3, 1.0)$, $(c_4, 0.02)$
H_2	$(c_4, 1.0)$

Using the same error functions and weight 1.0 for all constraints, we yield the following table of results for the two candidate valuations in example 1.

	$\theta = \theta_A$ $i = 1$	$\theta = \theta_B$ $i = 1$	$\theta = \theta_A$ $i = 2$	$\theta = \theta_B$ $i = 2$
$e(c_2\theta)\mu_i(c_2)$	0	0	0	0
$e(c_3\theta)\mu_i(c_3)$	0	0.01	0	0
$e(c_4\theta)\mu_i(c_4)$	0.018	0	0.9	0
$e(c_2\theta)^2\mu_i(c_2)$	0	0	0	0
$e(c_3\theta)^2\mu_i(c_3)$	0	0.0001	0	0
$e(c_4\theta)^2\mu_i(c_4)$	0.0162	0	0.81	0

If the *f-l-b* local comparator is used, we find that the two valuations are incomparable. Assuming a weight of 1.0 for all constraints in each level, the global comparators *f-w-s-b*, *f-w-c-b*, and *f-l-s-b* all conclude sales representative B to be the more desirable candidate for promotion.

In the new hierarchy, the constraint c_4 appears, besides in H_2, in H_1 with a membership of 0.02. In general, errors of c_2 and c_3 take precedence in the decision-making process since they both have a membership of 1.0. The influence of c_4 becomes effective only when the errors in c_4 for the candidate valuations are significant enough to overcome the suppression by the 0.02 membership factor. We use the same hierarchy to compare the valuations in example 2. The error of each constraint for each valuation and its product with the corresponding membership grades are summarized in the following tables:

	$\theta = \theta_A$	$\theta = \theta_B$
$e(c_1\theta)$	0	0
$e(c_2\theta)$	0	0
$e(c_3\theta)$	0	0.01
$e(c_4\theta)$	0.9	0.85

	$\theta = \theta_A$ $i = 1$	$\theta = \theta_B$ $i = 1$	$\theta = \theta_A$ $i = 2$	$\theta = \theta_B$ $i = 2$
$e(c_2\theta)\mu_i(c_2)$	0	0	0	0
$e(c_3\theta)\mu_i(c_3)$	0	0.01	0	0
$e(c_4\theta)\mu_i(c_4)$	0.018	0.017	0.9	0.85
$e(c_2\theta)^2\mu_i(c_2)$	0	0	0	0
$e(c_3\theta)^2\mu_i(c_3)$	0	0.0001	0	0
$e(c_4\theta)^2\mu_i(c_4)$	0.0162	0.01445	0.81	0.7225

Again, the local comparator finds the two valuations incomparable. The global comparator *f-w-s-b* recommends candidate A for promotion. The *f-w-c-b* comparator presents a biased view by looking at only the extreme case of a level. The *f-l-s-b* comparator amplifies error differences by comparing the square of the errors. In both cases, the errors of c_4 become more prominent in level H_1 and candidate B is recommended.

3.4 Remarks

Weight and membership grade of a constraint seem to be similar concepts and might be combined in the definition of the example global comparators. They are different for two reasons, one conceptual and one pragmatic. First, the weight of a constraint is a measure of the relative importance of the constraint as compared to other constraints within a level, whereas the membership grade of a constraint denotes the degree that the constraint belong to a level. Second, as can be seen from examples 1 and 2, the dominance problem persists no matter how we adjust the weights of constraints in the original constraint hierarchy: strong constraints still dominate over the weaker ones. On the other hand, membership grade allows a generally weak constraint to exercise its influence in stronger levels. Since the two concepts describe two different properties of a constraint with respect to a level, we suggest to keep them as two separate values instead of combining them into a single constant.

The relationship between local and global comparators in the CH framework carries forward to their fuzzy counterparts in the FCH framework. In other words, the solution set found using the fuzzy-locally-better comparator contains both of those found using the fuzzy-weighted-sum-better and the fuzzy-least-square-better comparators.

Proposition 1. *For a given error function e,*

$$\forall\theta\forall\sigma\forall H \;\; f\text{-}l\text{-}b(\theta,\sigma,H) \rightarrow f\text{-}w\text{-}s\text{-}b(\theta,\sigma,H)$$
$$\forall\theta\forall\sigma\forall H \;\; f\text{-}l\text{-}b(\theta,\sigma,H) \rightarrow f\text{-}l\text{-}s\text{-}b(\theta,\sigma,H).$$

Corollary 1. *For a given fuzzy constraint hierarchy, let S_{FLB} denote the set of solutions found using the f-l-b comparator, and S_{FWSB} and S_{FLSB} those found using the f-w-s-b and f-l-s-b comparators respectively. We have $S_{FWSB} \subseteq S_{FLB}$ and $S_{FLSB} \subseteq S_{FLB}$.*

We skip the proofs of the results since they are similar to those of the corresponding results for the CH framework found in [2]. As expected, the *f-l-b* comparator does not imply the *f-g-b* schema instantiated with an arbitrary combining function g. For example, the *f-l-b* comparator does not imply the *f-w-c-b* comparator.

4 Fuzzifying the Equality Relation

To equal or not to equal, that is the question.

Our second extension aims to solve the stiff equality problem. We formalize the notion of *approximately-equal-to* and define new valuation comparators using the modified equality relation. The α-approximate-equality relation $=_{a(\alpha)}$ is obtained by computing the α-cut [4] of the fuzzified equality relation, where α is a user-defined constant that controls the degree of approximation or tolerance.

A relation is a set of tuples. To fuzzify a crisp relation R, we need to devise a meaningful membership function that conforms to the intended interpretation of the relation. The membership function for the fuzzified relation, say R^f, should also provide a metric to measure how "well" an element tuple belongs to R^f. Thus the function must assign a membership of 1.0 to elements of R and the membership grade assigned to a non-member x of R should reflect the degree of x satisfying the property described by the intended interpretation of R. We are interested in only a subset of the standard equality relation $=$ since we only test the equality of non-negative real numbers. While many membership functions are possible, we propose to use the following membership function for the fuzzy relation $=^f(x, y)$:

$$\mu_{=^f}((x, y)) = \begin{cases} 1 & \text{if } x = y = 0 \\ 1 - \left|\frac{x-y}{x+y}\right| & \text{Otherwise} \end{cases}$$

for any non-negative real numbers x and y.

The function $\mu_{=^f}$ provides a measure on how close two real numbers are. The fuzzy relation $=^f$, however, cannot be used directly in decision-making. We need to extract the set of elements of $=^f$, the membership of which exceeds a certain user supplied *approximation threshold* $\alpha \in [0, 1]$, by computing the α-cut of $=^f$. An α-*cut* of a fuzzy set A is a crisp set A_α that contains all the elements of the universal set $\mathcal{D}$ that have a membership grade in A greater than or equal to the specified value of α, written as:

$$A_\alpha = \{x \in \mathcal{D} \mid \mu_A(x) \geq \alpha\}.$$

We define the α-*approximate equality relation* $=_{a(\alpha)}$ to be $=^f_\alpha$. We say that x is α-*approximately equal to* y, written $x =_{a(\alpha)} y$ (by abusing notation), if $(x, y) \in =_{a(\alpha)}$.

By replacing $=$ by $=_{a(\alpha)}$ in the definition of the *l-b* comparator and simple mathematics, we yield the *approximate-locally-better* comparator *a-l-b* as follows:

$$\begin{aligned} &a\text{-}l\text{-}b(\theta, \sigma, H, \alpha) \equiv \exists k > 0 \text{ such that} \\ &\quad \forall i \in \{1, \ldots, k-1\} \forall p \in H_i \; e(p\theta) =_{a(\alpha)} e(p\sigma) \\ &\quad \wedge \exists q \in H_k \; (e(q\theta) \neq_{a(\alpha)} e(q\sigma) \wedge e(q\theta) < e(q\sigma)) \\ &\quad \wedge \forall r \in H_k \; (e(r\theta) < e(r\sigma) \vee e(r\theta) =_{a(\alpha)} e(r\sigma)) \end{aligned}$$

The *approximate-globally-better* schema *a-g-b* is obtained by modifying the *g-b* schema in a similar fashion.

$$\begin{aligned} &a\text{-}g\text{-}b(\theta, \sigma, H, g, \alpha) \equiv \exists k > 0 \text{ such that} \\ &\quad \forall i \in \{1, \ldots, k-1\} \; g(\theta, H_i) =_{a(\alpha)} g(\sigma, H_i) \\ &\quad \wedge (g(\theta, H_k) \neq_{a(\alpha)} g(\sigma, H_k) \wedge g(\theta, H_k) < g(\sigma, H_k)) \end{aligned}$$

We get three global comparators *approximate-weighted-sum-better a-w-s-b*, *approximate-worst-case-better a-w-c-b*, and *approximate-least-square-better a-l-s-b* by instantiating the *a-g-b* schema with the same combining functions g for their classical counterparts. We call this class of comparators *approximate comparators*.

4.1 Illustrative Examples

In the following, we show how the new comparators can be used to solve the stiff equality problem as exemplified in examples 3 and 4. Note that the new comparators have no connection to the FCH framework (yet). In the following, we assume that the sales representative promotion problem is modeled using the CH framework as in example 1. We first give a table containing the membership grade of the number pairs for the fuzzy relation $=^f$ needed for our further discussion.

x	y	$\mu_{=^f}(x, y)$
0.42	0.4	0.976
0.31	0.3	0.984
$0.72\ (= 0.42 + 0.3)$	$0.71\ (= 0.4 + 0.31)$	0.993
$0.73\ (= 0.42 + 0.31)$	$0.7\ (= 0.4 + 0.3)$	0.979
$0.2664\ (= 0.42^2 + 0.3^2)$	$0.2561\ (= 0.4^2 + 0.31^2)$	0.980
$0.2725\ (= 0.42^2 + 0.31^2)$	$0.25\ (= 0.4^2 + 0.3^2)$	0.957

Suppose we set the approximation threshold α to 0.95. All approximate comparators defined are able to conclude that the valuations for both examples are equivalent at level H_1, leaving the decision to level H_2. Therefore, candidate A is recommended.

The threshold α can be adjusted to the user's liking and need. Suppose now that α is 0.98. The following table summarizes the recommendation made by each of the approximate comparators for both scenarios.

Example	*a-l-b*	*a-w-s-b*	*a-w-c-b*	*a-l-s-b*
3	$B\ (H_1)$	$A\ (H_2)$	$B\ (H_1)$	$A\ (H_2)$
4	$B\ (H_1)$	$B\ (H_1)$	$B\ (H_1)$	$B\ (H_1)$

Each row contains the recommendations for each problem scenario; while each column corresponds to recommendations from each approximate comparator. We also record the level at which the decision is made in brackets beside each recommendation. By increasing the threshold, the equality becomes more stiff and more undesirable recommendations are made using information solely in level H_1.

4.2 Remarks

The threshold α offers fine control over the stiffness of the α-approximate equality relation. On one end of the spectrum, when $\alpha = 1$, the α-approximate equality relation is reduced to the standard equality relation. All approximate comparators are also reduced to the original comparators. On the other end of the spectrum, when $\alpha = 0$, any pair of error values are considered to be α-approximately equal to each other. In this case, all valuations are incomparable no matter which comparator is used. The solution set is always S_0, which is the set of solutions for the required constraints. In other words, the soft constraints have no effect in the calculation of the solution set.

As shown in the rework of example 3 with $\alpha = 0.98$, the comparator *a-l-b* implies neither *a-w-s-b* nor *a-l-s-b* in general. Thus, solution sets computed from the new comparators do not inherit the solution set subsumption properties of their classical counterparts.

The proposed membership function $\mu_{=^f}((x, y))$ for the fuzzy relation $=^f$ aims to measure and compare the order of magnitude of the difference between x and y against that of the sum of x and y. While this measure works well in general, it can give results that may be counter-intuitive to some. For example, the function will assign the same membership grade to the pairs $(0.02, 0.03)$, $(2, 3)$, and even $(2.0 \times 10^6, 3.0 \times 10^6)$. The phenomenon is due to the fact that the difference between the two numbers in each pair all appears in the same significant digit, in this case the first digit, and the digits differ by the same amount. In addition, the membership grade of $(x, 0)$ is always 0 for all $x > 0$. Thus, even $(1.0 \times 10^{-9}, 0)$ will have a membership grade of 0. This is because the difference between any quantity and void is infinite. From being nothing to something does constitute a quantum jump. If one finds this membership function not suitable, other membership functions, such as,

$$\mu_{=^f}((x, y)) = 1 - \left| \frac{x - y}{x + y + c} \right|, \text{ for some constant } c > 0$$

$$\mu_{=^f}((x, y)) = \begin{cases} |x - y| & \text{if } |x - y| < 1 \\ 1 - \left| \frac{x-y}{x+y} \right| & \text{Otherwise} \end{cases}$$

are possible as long as the function provides a reasonable measure of how similar two nonnegative numbers are.

5 Combining the Frameworks

The FCH framework and the approximate comparators address different aspects of the CH framework. The fuzzy comparators still suffer from the stiff equality problem and the approximate comparators do not solve the dominance problem. Since the two generalizations complement each other, the next natural step is to incorporate the α-approximate equality relation into the fuzzy comparators, yielding a new class of fuzzy comparators. We use the same naming convention for the comparators. Their abbreviated names should be self-explanatory.

The straight-forward generalization of the fuzzy local comparator follows.

$$\begin{aligned} &a\text{-}f\text{-}l\text{-}b(\theta, \sigma, H, \alpha) \equiv \exists k > 0 \text{ such that} \\ &\quad \forall i \in \{1, \ldots, k-1\} \forall (p, \mu_i(p)) \in H_i \;\; e(p\theta)\mu_i(p) =_{a(\alpha)} e(p\sigma)\mu_i(p) \\ &\quad \wedge \exists (q, \mu_k(q)) \in H_k \;\; (e(q\theta)\mu_k(q) \neq_{a(\alpha)} e(q\sigma)\mu_k(q) \;\; \wedge e(q\theta)\mu_k(q) < e(q\sigma)\mu_k(q)) \\ &\quad \wedge \forall (r, \mu_k(r)) \in H_k \;\; (e(r\theta)\mu_k(r) < e(r\sigma)\mu_k(r) \;\; \vee e(r\theta)\mu_k(r) =_{a(\alpha)} e(r\sigma)\mu_k(r)) \end{aligned}$$

This definition should explain why the terms $\mu_i(p)$, $\mu_k(q)$, and $\mu_k(r)$ are essential: they do not cancel out under the α-approximate equality relation.

The definitions of the global schema follows.

$$\begin{aligned}
&a\text{-}f\text{-}g\text{-}b(\theta, \sigma, H, g, \alpha) \equiv \exists k > 0 \text{ such that} \\
&\quad \forall i \in \{1, \ldots, k-1\}\ g(\theta, H_i, \mu_i) =_{a(\alpha)} g(\sigma, H_i, \mu_i) \\
&\wedge (g(\theta, H_k, \mu_k) \neq_{a(\alpha)} g(\sigma, H_k, \mu_k) \wedge g(\theta, H_k, \mu_k) < g(\sigma, H_k, \mu_k))
\end{aligned}$$

We get three global comparators *a-f-w-s-b*, *a-f-w-c-b*, and *a-f-l-s-b* by instantiating the global schema with the same combining functions g for their fuzzy counterparts. We call this class of operators *combined operators*.

Assuming an approximation threshold of $\alpha = 0.95$, we obtain the following tables of recommendations from each of the combined comparators for the scenarios described in examples 1 to 4.

Example	*a-f-l-b*	*a-f-w-s-b*	*a-f-w-c-b*	*a-f-l-s-b*
1	A, B (H_1)	B (H_1)	B (H_1)	B (H_1)
2	A (H_1)	A (H_1)	A, B (H_2)	B (H_1)
3	A (H_1)	A (H_2)	A (H_2)	A (H_2)
4	A (H_1)	A (H_2)	A (H_2)	A (H_2)

The level at which the recommendation is made is enclosed in brackets. The recommendation "A, B" means that the two valuations are incomparable. The recommendations made by the combined comparators are at least as good as and, in some cases, better than the ones made by either class of the previously defined new comparators.

6 Concluding Remarks

The contribution of this paper is four-fold. First, we review the CH framework and use realistic examples to illustrate the dominance problem and the stiff equality problem. We also pinpoint aspects of the framework for further enhancement. Second, we propose the FCH framework to solve the dominance problem. In the new framework, each constraint can belong to more than one level of a hierarchy and to a varying degree of membership. New local and global comparators, which take the membership grade of constraints into account, are introduced. Following the style of [2], we also show the formal properties and relationship of the new comparators. Third, we give a fuzzy version $=^f$ of the equality relation and define the α-approximate equality relation to be the α-cut of $=^f$. Another new class of comparators is defined based on the α-approximate equality for solving the stiff equality problem. Fourth, we show a seamless integration of the the α-approximate equality relation and the FCH framework. The combined framework allows more flexibility in the modeling and handling of MCS problems. In particular, the solutions computed from the combined framework correspond more to how human makes decision, at least for the four example problems presented.

Our approach in adopting fuzzy techniques in constraint research differs from that of previous work (such as [1]) in that we fuzzify, instead of constraints,

(1) the level membership of constraints and (2) the equality relation used in *better* comparators. Further work is needed to study the connection between our framework and the closely related Multiple Attribute (or Criteria or Objective) Decision-Making (MADM) methods in the fuzzy system [6] and operations research [8] literature. As with other fuzzy-based systems, experimentation and experience are needed to determine the membership grades of constraints and the approximation threshold α for a particular application. The efficient knowledge elicitation technique by Slany [7] and the learning technique by Rossi and Sperduti [5] are good starting points.

We have demonstrated how the FCH framework can be used to better model a large class of over-constrained problems, namely those in MCS. More experience in using the FCH framework is needed to understand how the framework can help better tackle other over-constrained problems. There is also need to study if the FCH framework exhibits undesirable behavior in other problem instances.

Two implementations of FCH for the finite domain are available, one in spreadsheet and one using C. A more sophisticated implementation in Prolog of the combined framework for the finite domain is in progress. Existing solution algorithms for constraint hierarchies should also be easily adapted for solving fuzzy constraint hierarchies since the difference between the two frameworks lies mainly in the definition of the comparators.

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