ENGG1410-F
Tutorial 1

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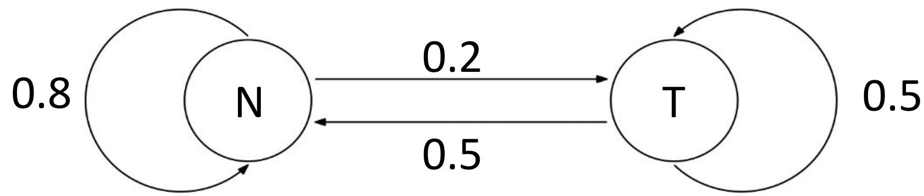
Production Problem

- In a production process, let N mean "no trouble" and T "trouble." Let the transition probabilities from one day to the next be 0.8 for $N \to N$, hence 0.2 for $N \to T$, and 0.5 for $T \to N$, hence 0.5 for $T \to T$.
- If on the first day there is no trouble(N), what is the probability of N on the second today? How about the 10th day?

From Page 271, Problem 26 of the textbook

- The state transition process can be modeled as the following graph:



- It can also be represented as a Matrix $A = [a_{ij}]$. Call N as "state 1", and T as "state 2". a_{ij} is the probability of moving from state i to state j.

$$\mathrm{A} = \begin{bmatrix} 0.8 & 0.2 \\ 0.5 & 0.5 \end{bmatrix}$$

- The probabilities of N and T on the i-th day form a column vector x_i. Initially,

$$x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

- You can verify that the probabilities of N and T on the second day can be derived as

$$x_2 = A^T x_1 = \begin{bmatrix} 0.8 \\ 0.2 \end{bmatrix}$$

- The probabilities of N and T on the 10-th day can be computed as

$$x_{10} = (A^T)^9 \cdot x_1$$

- A fast way to compute $(A^T)^9$ is as follows
  - $(A^T)^2 = A^T A^T$
  - $(A^T)^4 = (A^T)^2 (A^T)^2$
  - $(A^T)^8 = (A^T)^4 (A^T)^4$
  - $(A^T)^9 = (A^T)^8 A^T$

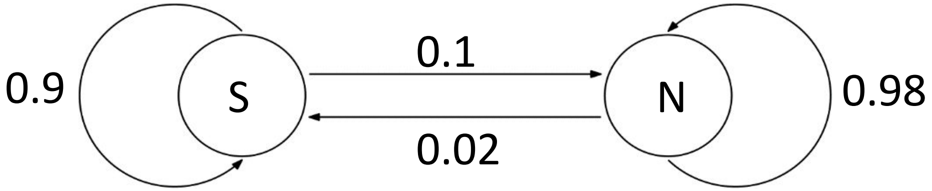
Concert subscription

- In a community of 10,000 adults, there are some subscribers to a concert series and non-subscribers.
- Subscribers tend to renew their subscription in the next year with probability 0.9.
- Persons presently not subscribing will subscribe for the next year with probability 0.02.
- If the present number of subscribers is 2000, how many subscribes there would be in the 10-th year in expectation?

From Page 271, Problem 28 of the textbook

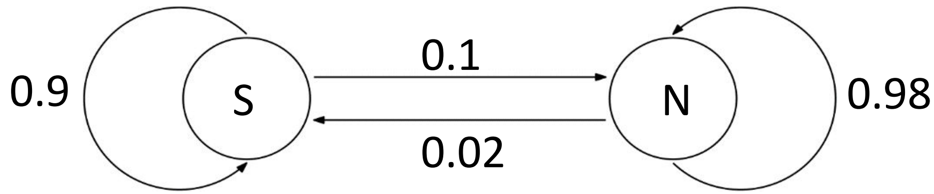
- Key observation: analyze each person separately.

- Model the state transition of a person as a graph:



- It can also be represented as a Matrix $A = [a_{ij}]$:

$$A = \begin{bmatrix} 0.9 & 0.1 \\ 0.02 & 0.98 \end{bmatrix}$$

- 

- A subscriber has an initial vector:

$$x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

- Similarly, a non-subscriber has an initial vector

$$x_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- For each person, her/his probability vector in the 10-th year can be obtained from

$$x_{10} = (A^T)^9 \cdot x_1$$

- Let a be the number at the first row of:

$$(A^T)^9 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

  - Note that a is the probability that a current subscriber remains as a subscriber in the 10-th year.

- Let b be the number at the first row of:

$$(A^T)^9 \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

  - Note that b is the probability that a current non-subscriber is a subscriber in the 10-th year.

- The expected number of subscribers in the 10-th year is

$$2000 \cdot a + (10000 - 2000) \cdot b$$

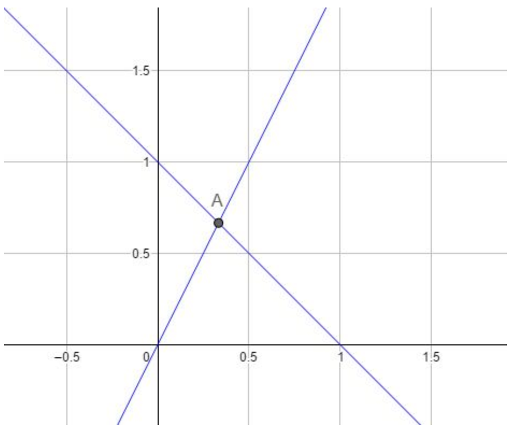
Solutions to a linear system – a geometric perspective

- Given a linear system with m equations about n variables.
- For simplicity, consider $m = n = 2$, i.e., there are two equations and two unknown variables x and y.
- If we interpret x and y as coordinates in the plane, then each equation represents a straight line.
- $(x, \ y)$ is a solution if and only if the point *p* = $(x, \ y)$ lies on both lines.

From Page 273 of the textbook

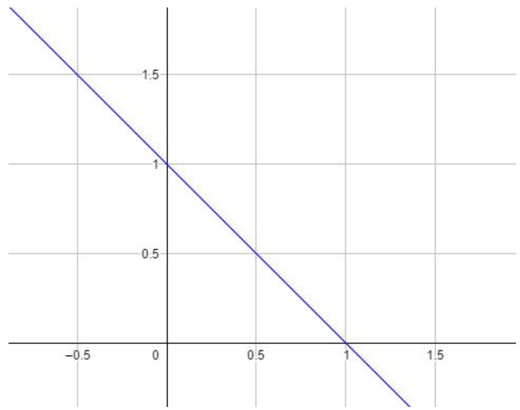
Three kinds of solutions to a linear system

- For instance.



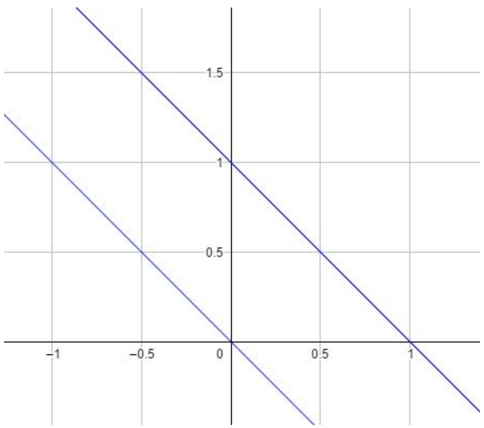
$$x + y = 1$$
$$2x - y = 0$$

Case 1. Unique Solution



$$x + y = 1$$
$$2x + 2y = 2$$

Case 2. Infinite Solutions



$$x + y = 1$$
$$x + y = 0$$

Case 3. No Solution

Row Elementary Matrix

- Row elementary operations can be accomplished by matrix multiplications.
- Suppose that A is an $m \times n$ matrix. If M is obtained from A by an elementary row operation, then there is always an $m \times m$ matrix E such that,

$$M = EA$$

- Such an E is called an *elementary matrix*.

From Page 281, Problem 24 of the textbook

Row Elementary Matrix

- For example，
- $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$
- $E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix},$

- $E_1 A \leftrightarrow$ Switch the first two rows.
- $E_2 A \leftrightarrow$ Multiply the second row of A by 2.
- $E_3 A \leftrightarrow$ Add the 2^{nd} row into the 3^{rd} row.

Row Elementary Matrix

- Suppose that A is an $m \times n$ matrix. If M is obtained from A by an elementary row operation, then there is an $m \times m$ elementary matrix E such that,

$$M = EA$$

- Interestingly, a convenient way to obtain E is to perform the same row operation on the $m \times m$ identity matrix.
  - See the next few slides for examples.

Row Elementary Matrix

- $$M = EA$$
- A convenient way to obtain E is to perform the same row operation on the $m \times m$ identity matrix.
- Suppose that we want to switch the first two rows of $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$.
- Performing the same operation on the 3×3 identity matrix $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ yields $E = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. This is the elementary matrix for the above operation.

Row Elementary Matrix

- $$M = EA$$
- A convenient way to obtain E is to perform the same row operation on the $m \times m$ identity matrix.
- Suppose that we want to switch the multiply the second row of $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ by 2.
- Performing the same operation on the 3×3 identity matrix $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ yields $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. This is the elementary matrix for the above operation.

Row Elementary Matrix

- $$M = EA$$
- A convenient way to obtain E is to perform the same row operation on the $m \times m$ identity matrix.
- Suppose that we want to add up the last two rows of $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$, and use the result to replace the 3rd row.
- Performing the same operation on the 3×3 identity matrix $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ yields $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$. This is the elementary matrix for the above operation.