Question 1.
Solution.

$$\begin{bmatrix} 1 & 4 & 1 & 1 & 1 \\ -1 & -3 & -2 & 1 & 1 \\ 2 & 0 & 6 & 1 & 1 \\ 0 & 2 & -1 & 2 & 1 \\ 0 & 2 & -1 & 2 & 1 \end{bmatrix} \tag{1}$$

$$\rightarrow \begin{bmatrix} 1 & 4 & 1 & 1 & 1 \\ 0 & 1 & -1 & 2 & 2 \\ 0 & -8 & 4 & -1 & -1 \\ 0 & 2 & -1 & 2 & 1 \\ 0 & 2 & -1 & 2 & 1 \end{bmatrix} \begin{matrix} \\ Row2 + Row1 \\ Row3 - 2Row1 \\ \\ \\ \end{matrix} \tag{2}$$

$$\rightarrow \begin{bmatrix} 1 & 4 & 1 & 1 & 1 \\ 0 & 1 & -1 & 2 & 2 \\ 0 & 0 & -4 & 15 & 15 \\ 0 & 0 & 1 & -2 & -3 \\ 0 & 0 & 1 & -2 & -3 \end{bmatrix} \begin{matrix} \\ \\ Row3 + 8Row2 \\ Row4 - 2Row2 \\ Row5 - 2Row2 \end{matrix} \tag{3}$$

$$\rightarrow \begin{bmatrix} 1 & 4 & 1 & 1 & 1 \\ 0 & 1 & -1 & 2 & 2 \\ 0 & 0 & -4 & 15 & 15 \\ 0 & 0 & 0 & 7 & 3 \\ 0 & 0 & 0 & 7 & 3 \end{bmatrix} \begin{matrix} \\ \\ \\ 4Row4 + Row3 \\ 4Row5 + Row3 \end{matrix} \tag{4}$$

$$\rightarrow \begin{bmatrix} 1 & 4 & 1 & 1 & 1 \\ 0 & 1 & -1 & 2 & 2 \\ 0 & 0 & -4 & 15 & 15 \\ 0 & 0 & 0 & 7 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} \\ \\ \\ \\ Row5 - Row4 \end{matrix} \tag{5}$$

Hence, the rank of the matrix is 4.

Question 2.

Solution.

(a) Augmented matrix

$$\begin{bmatrix} 2 & 4 & -2 & 0 \\ 3 & 5 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 6 & 12 & -6 & 0 \\ 6 & 10 & 0 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 6 & 12 & -6 & 0 \\ 0 & -2 & 6 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & -3 & -1 \end{bmatrix}$$
$$\Rightarrow \begin{bmatrix} 1 & 0 & 5 & 2 \\ 0 & 1 & -3 & -1 \end{bmatrix}$$

This indicates equations:

$$\begin{aligned} x_1 + 5x_3 &= 2 \\ x_2 - 3x_3 &= -1 \end{aligned}$$

Hence all the solutions $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ constitute the set:

$$\left\{ \begin{bmatrix} 2 - 5t \\ -1 + 3t \\ t \end{bmatrix} \middle| \; t \in \mathbb{R} \right\}$$

(b) Augmented matrix

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 2 & 1 & 4 & 9 \\ 3 & -1 & a & b \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & -1 & 2 & -3 \\ 0 & -4 & a-3 & b-18 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & -1 & 2 & -3 \\ 0 & 0 & a-11 & b-6 \end{bmatrix}$$

Hence, for the system to have a unique solution, it must hold that $a \neq 11$ (the value of b can be anything).

(c) For the system to have no solutions, it must hold that $a = 11$ and $b \neq 6$.

Question 3.

Solution.

(a) It is easy to show that $det(\boldsymbol{A}) = det(\boldsymbol{B}) = 36$. Therefore, $det(\boldsymbol{A}^{-1}\boldsymbol{B}^T) = \frac{1}{det(\boldsymbol{A})} \cdot det(\boldsymbol{B}) = 1$.

(b)

$$
\begin{aligned}
x_1 &= \frac{\begin{vmatrix} 9 & -1 & 0 \\ 5 & 1 & 1 \\ 8 & 1 & -1 \end{vmatrix}}{\begin{vmatrix} 1 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & -1 \end{vmatrix}} = \frac{-31}{-1} = 31 \\
x_2 &= \frac{\begin{vmatrix} 1 & 9 & 0 \\ -1 & 5 & 1 \\ 0 & 8 & -1 \end{vmatrix}}{\begin{vmatrix} 1 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & -1 \end{vmatrix}} = \frac{-22}{-1} = 22 \\
x_3 &= \frac{\begin{vmatrix} 1 & -1 & 9 \\ -1 & 1 & 5 \\ 0 & 1 & 8 \end{vmatrix}}{\begin{vmatrix} 1 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & -1 \end{vmatrix}} = \frac{-14}{-1} = 14
\end{aligned}
$$

Question 4.

Solution.

1. Compute A^{-1}: The augmented matrix is expressed as

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 3 & 3 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \; row2' = row2 - row1;$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 3 & 3 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \; row3' = row3 - row1; \; row3' = row3 - 2row2$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & -2 & 1 & 0 \\ 1 & 3 & 3 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \; row4' = row4 - row1 - 3row2 - 3row3$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 3 & -3 & 1 \end{bmatrix}$$

$$\Rightarrow \; A^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix}$$

2. Compute $(\frac{1}{2}A)^{-1}$:

$$(\frac{1}{2}A)^{-1} = 2A^{-1} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ -2 & 2 & 0 & 0 \\ 2 & -4 & 2 & 0 \\ -2 & 6 & -6 & 2 \end{bmatrix}.$$

Question 5.

Solution. Consider the characteristic equation of $\boldsymbol{A}$:

$$\begin{aligned} |\boldsymbol{A} - \lambda \boldsymbol{I}| &= 0 \\ \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} &= 0 \\ (a-\lambda)(d-\lambda) - bc &= 0 \\ \lambda^2 - (a+d)\lambda + ad - bc &= 0. \end{aligned}$$

Hence

$$\begin{aligned} \lambda_1 &= \frac{(a+d) + \sqrt{\Delta}}{2} \\ \lambda_2 &= \frac{(a+d) - \sqrt{\Delta}}{2} \end{aligned}$$

where $\Delta = (a+d)^2 - 4(ad - bc)$. It thus follows that $\lambda_1 + \lambda_2 = a + d$.

Question 6.

Solution. It is clear that $\boldsymbol{A}$ has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 0$. We aim to find three eigenvectors that are linearly independent.

Towards this purpose, let us first obtain the eigenspace of λ_1, i.e., the set of $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ satisfying

$$\begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0.$$

This set can be represented as

$$\left\{ \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} \,\middle|\, t \in \mathbb{R} \right\}$$

Pick an arbitrary non-zero vector from the set, e.g., $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

Next, let us obtain the eigenspace of λ_2, i.e., the set of $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ satisfying

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0.$$

This set can be represented as

$$\left\{ \begin{bmatrix} v \\ u \\ v \end{bmatrix} \,\middle|\, u, v \in \mathbb{R} \right\}$$

Pick two non-zero vectors that are linearly independent, e.g., $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

With the above, we can decide $\boldsymbol{P}$ and $\boldsymbol{B}$ as:

$$\boldsymbol{P} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Question 7.
Solution.

(a) Since $\boldsymbol{A}$ is skew-symmetric, we have

$$\begin{aligned} a &= d = 0 \\ c &= -b \end{aligned}$$

Since $\boldsymbol{A}$ is orthogonal, we have

$$\boldsymbol{A}^T = \boldsymbol{A}^{-1}$$

Consider

$$\begin{gathered} \boldsymbol{I} = \boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{A}\boldsymbol{A}^T \quad \Rightarrow \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix} \begin{bmatrix} 0 & -b \\ b & 0 \end{bmatrix} = \begin{bmatrix} b^2 & 0 \\ 0 & b^2 \end{bmatrix} \quad \Rightarrow \\ b = \pm 1 \end{gathered}$$

Therefore,

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

(b) If $\boldsymbol{B}$ is skew-symmetric, then

$$e = 0$$

This implies

$$\det(\boldsymbol{B}) = 0$$

which means that $\boldsymbol{B}^{-1}$ does not exist. Therefore, $\boldsymbol{B}$ is not orthogonal.