# Lecture Notes: Determinant of a Square Matrix

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## 1 Determinant Definition

Let $\boldsymbol{A} = [a_{ij}]$ be an $n \times n$ matrix (i.e., $\boldsymbol{A}$ is a square matrix). Given a pair of $(i, j)$, we define $\boldsymbol{A}_{ij}$ to be the $(n-1) \times (n-1)$ matrix obtained by removing the $i$-th row and $j$-th column of $\boldsymbol{A}$. For example, suppose that

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 0 & -2 \\ -1 & -1 & 2 \end{bmatrix}$$

Then:

$$\boldsymbol{A}_{21} = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}, \boldsymbol{A}_{22} = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}, \boldsymbol{A}_{32} = \begin{bmatrix} 1 & 1 \\ 3 & -2 \end{bmatrix}$$

We are now ready to define determinants:

**Definition 1.** *Let $\boldsymbol{A} = [a_{ij}]$ be an $n \times n$ matrix. If $n = 1$, its* **determinant**, *denoted as $det(\boldsymbol{A})$, equals $a_{11}$. If $n > 1$, we first choose an arbitrary $i^* \in [1, n]$, and then define the determinant of $\boldsymbol{A}$ recursively as:*

$$det(\boldsymbol{A}) = \sum_{j=1}^{n} (-1)^{i^*+j} \cdot a_{i^*j} \cdot det(\boldsymbol{A}_{i^*j}). \tag{1}$$

Besides $det(\boldsymbol{A})$, we may also denote the determinant of $\boldsymbol{A}$ as $|\boldsymbol{A}|$. Henceforth, if we apply (1) to compute $det(\boldsymbol{A})$, we say that we *expand* $\boldsymbol{A}$ *by row* $i^*$. It is important to note that the value of $det(\boldsymbol{A})$ does *not* depend on the choice of $i^*$. We omit the proof of this fact, but illustrate it in the following examples.

**Example 1 (Second-Order Determinants).** In general, if $\boldsymbol{A} = [a_{ij}]$ is a $2 \times 2$ matrix, then

$$det(\boldsymbol{A}) = a_{11}a_{22} - a_{12}a_{21}.$$

For instance:

$$\begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} = 2 \times 2 - 1 \times (-1) = 5.$$

We may verify the above by definition as follows. Choosing $i^* = 1$, we get:

$$\begin{aligned} \begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} &= (-1)^{1+1} \cdot 2 \cdot det(\boldsymbol{A}_{11}) + (-1)^{1+2} \cdot 1 \cdot det(\boldsymbol{A}_{12}) \\ &= 2 \times 2 + (-1) \times (-1) = 5. \end{aligned}$$

Alternatively, choosing $i^* = 2$, we get:

$$\begin{vmatrix} 2 & 1 \\ \text{-1} & 2 \end{vmatrix} = (-1)^{2+1} \cdot (-1) \cdot det(\boldsymbol{A}_{21}) + (-1)^{2+2} \cdot 2 \cdot det(\boldsymbol{A}_{22})$$
$$= 1 \times 1 + 2 \times 2 = 5.$$

□

**Example 2 (Third-Order Determinants).** Suppose that

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 0 & \text{-2} \\ \text{-1} & \text{-1} & 2 \end{bmatrix}$$

Choosing $i^* = 1$, we get:

$$\begin{aligned} det(\boldsymbol{A}) &= 1 \begin{vmatrix} 0 & -2 \\ -1 & 2 \end{vmatrix} - 2 \begin{vmatrix} 3 & -2 \\ -1 & 2 \end{vmatrix} + 1 \begin{vmatrix} 3 & 0 \\ -1 & -1 \end{vmatrix} \\ &= 1(0-2) - 2(6-2) + 1(-3-0) = -13. \end{aligned}$$

Alternatively, choosing $i^* = 2$, we get:

$$\begin{aligned} det(\boldsymbol{A}) &= -3 \begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} + 0 \begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix} - (-2) \begin{vmatrix} 1 & 2 \\ -1 & -1 \end{vmatrix} \\ &= (-3)(4+1) + 0(2+1) + 2(-1+2) = -13. \end{aligned}$$

□

## 2 Properties of Determinants

**Expansion by a Column.** Definition 1 allows us to compute the determinant of a matrix by row expansion. We may also achieve the same purpose by column expansion.

**Lemma 1.** *Let $\boldsymbol{A} = [a_{ij}]$ be an $n \times n$ matrix with $n > 1$. Choose an arbitrary $j^* \in [1, n]$. The determinant of $\boldsymbol{A}$ equals:*

$$det(\boldsymbol{A}) = \sum_{i=1}^{n} (-1)^{i+j^*} \cdot a_{ij^*} \cdot det(\boldsymbol{A}_{ij^*}).$$

*The value of the above equation does not depend on the choice of $j^*$.*

We omit the proof but illustrate the lemma with an example below. Henceforth, if we compute $det(\boldsymbol{A})$ by the above lemma, we say that we *expand $\boldsymbol{A}$ by column $j^*$.*

**Example 3.** Suppose that

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 0 & \text{-2} \\ \text{-1} & \text{-1} & 2 \end{bmatrix}$$

Choosing $j^* = 1$, we get:

$$\begin{aligned} det(\boldsymbol{A}) &= 1\begin{vmatrix} 0 & -2 \\ -1 & 2 \end{vmatrix} - 3\begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} + (-1)\begin{vmatrix} 2 & 1 \\ 0 & -2 \end{vmatrix} \\ &= 1(0-2) - 3(4+1) - 1(-4-0) = -13. \end{aligned}$$

□

**Corollary 1.** *Let $\boldsymbol{A}$ be a square matrix. Then, $det(\boldsymbol{A}) = det(\boldsymbol{A}^T)$.*

*Proof.* Note that expanding $\boldsymbol{A}$ by column $k$ is equivalent to expanding $\boldsymbol{A}^T$ by row $k$. □

**Corollary 2.** *If $\boldsymbol{A}$ has a zero row or a zero column, then $det(\boldsymbol{A}) = 0$.*

*Proof.* If $\boldsymbol{A}$ has a zero row, then $det(\boldsymbol{A}) = 0$ follows from expanding $\boldsymbol{A}$ by that row. The case where $\boldsymbol{A}$ has a zero column is similar. □

**Determinant of a Row-Echelon Matrix.** The next lemma shows that the determinant of a matrix in row-echelon form is simply the product of the values on the main diagonal.

**Lemma 2.** *Let $\boldsymbol{A} = [a_{ij}]$ be an $n \times n$ matrix in row-echelon form. Then, $det(\boldsymbol{A}) = \Pi_{i=1}^n a_{ii}$.*

*Proof.* We can prove the lemma by induction. First, correctness is obvious for $n = 1$. Assuming correctness for $n \le t-1$ (for $t \ge 2$), consider $n = t$. Expanding $\boldsymbol{A}$ by the first row gives:

$$det(\boldsymbol{A}) = \sum_{j=1}^{n} (-1)^{1+j} \cdot a_{1j} \cdot det(\boldsymbol{A}_{1j}). \tag{2}$$

From induction we know that $det(\boldsymbol{A}_{11}) = \Pi_{i=2}^n a_{ii}$. Furthermore, for $j > 1$, $det(\boldsymbol{A}_{1j}) = 0$ because the first column of $\boldsymbol{A}_{1j}$ contains all 0's. It thus follows that (2) equals $\Pi_{i=1}^n a_{ii}$. □

**Determinants under Elementary Row Operations.** Let $\boldsymbol{A} = [a_{ij}]$ be an $n \times n$ matrix. Recalled that the elementary row operations on $\boldsymbol{A}$ are:

1. Switch two rows of $\boldsymbol{A}$.
2. Multiply all numbers of a row of $\boldsymbol{A}$ by the same non-zero value $c$.
3. Let $\boldsymbol{r}_i$ and $\boldsymbol{r}_j$ be two distinct row vectors of $\boldsymbol{A}$. Update row $\boldsymbol{r}_i$ to $\boldsymbol{r}_i + \boldsymbol{r}_j$.

Next, we refer to the above as Operation 1, 2, and 3, respectively.

**Lemma 3.** *The determinant of $\boldsymbol{A}$*

1. *should be multiplied by $-1$ after Operation 1;*
2. *should be multiplied by $c$ after Operation 2;*
3. *has no change after Operation 3.*

*Proof.* See appendix. □

The following corollary will be very useful:

**Corollary 3.** *Let $\boldsymbol{r}_i$ and $\boldsymbol{r}_j$ be two distinct row vectors of $\boldsymbol{A}$. The determinant of $\boldsymbol{A}$ does not change after the following operation:*

- *Update row $\boldsymbol{r}_i$ to $\boldsymbol{r}_i + c \cdot \boldsymbol{r}_j$, where $c$ is a real value.*

*Proof.* We consider only $c \neq 0$ (the case of $c = 0$ is trivial). Let $\boldsymbol{A'}$ be the array after applying the above operation. We can also obtain $\boldsymbol{A'}$ by performing the next three operations:

1. Multiply the $j$-th row of $\boldsymbol{A}$ by $c$. Let $\boldsymbol{A_1}$ be the array obtained.
2. Add the $j$-th row of $\boldsymbol{A_1}$ into its $i$-th row. Let $\boldsymbol{A_2}$ be the array obtained.
3. Multiply the $j$-th row of $\boldsymbol{A_2}$ by $1/c$. Let $\boldsymbol{A_3}$ be the array obtained. Note that $\boldsymbol{A_3} = \boldsymbol{A'}$.

By Lemma 3, $det(\boldsymbol{A_1}) = c \cdot det(\boldsymbol{A})$, $det(\boldsymbol{A_2}) = det(\boldsymbol{A_1})$, and $det(\boldsymbol{A_3}) = (1/c) \cdot det(\boldsymbol{A_3})$. Hence, $det(\boldsymbol{A}) = det(\boldsymbol{A'})$. □

Let us illustrate Lemma 3 and Corollary 3 with an example.

**Example 4.**

$$\begin{vmatrix} 1 & 2 & 1 \\ 3 & 0 & -2 \\ -1 & -1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 1 \\ 3 & 0 & -2 \\ 0 & 1 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 1 \\ 0 & -6 & -5 \\ 0 & 1 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 1 \\ 0 & -6 & -5 \\ 0 & 0 & 13/6 \end{vmatrix} = -13.$$

Here is another derivation giving the same result:

$$\begin{vmatrix} 1 & 2 & 1 \\ 3 & 0 & -2 \\ -1 & -1 & 2 \end{vmatrix} = - \begin{vmatrix} 1 & 2 & 1 \\ -1 & -1 & 2 \\ 3 & 0 & -2 \end{vmatrix} = - \begin{vmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & -6 & -5 \end{vmatrix} = - \begin{vmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 13 \end{vmatrix} = -13.$$

□

**Corollary 4.** *If $\boldsymbol{A}$ has two identical rows or columns, then $det(\boldsymbol{A}) = 0$.*

*Proof.* We prove only the row case. Switching the two rows gets back the same matrix. However, by Lemma 3, the determinant of the matrix should be multiplied by $-1$. Therefore, we get $det(\boldsymbol{A}) = -det(\boldsymbol{A})$, meaning $det(\boldsymbol{A}) = 0$. □

**Determinant under Matrix Multiplication.** The following is a perhaps surprising property of determinants:

**Lemma 4.** *Let $\boldsymbol{A}, \boldsymbol{B}$ be $n \times n$ matrices. It holds that $det(\boldsymbol{AB}) = det(\boldsymbol{A}) \cdot det(\boldsymbol{B})$.*

The proof is not required, but we will discuss it in a tutorial after we have learned the concept of "matrix inversion".

**Example 5.**

$$\begin{vmatrix} 1 & 2 & 1 \\ 3 & 0 & -2 \\ -1 & -1 & 2 \end{vmatrix} = -13$$

$$\begin{vmatrix} -2 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & -2 & 0 \end{vmatrix} = -3$$

$$\left| \begin{bmatrix} 1 & 2 & 1 \\ 3 & 0 & -2 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} -2 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & -2 & 0 \end{bmatrix} \right| = \begin{vmatrix} -1 & 1 & 2 \\ -8 & 7 & 0 \\ 4 & -6 & -1 \end{vmatrix} = 39.$$

□

**Relationships with Ranks.** The lemma below relates determinants to ranks:

**Lemma 5.** *Let $\boldsymbol{A}$ be an $n \times n$ matrix. $\boldsymbol{A}$ has rank $n$ if and only if $det(\boldsymbol{A}) \neq 0$.*

*Proof.* We can first apply elementary row operations to convert $\boldsymbol{A}$ into row-echelon form $\boldsymbol{A}^*$. Thus, $\boldsymbol{A}$ has rank $n$ if and only if $\boldsymbol{A}^*$ has rank $n$. Since $\boldsymbol{A}^*$ is a square matrix, that it has rank $n$ is equivalent to saying that all the numbers on its main diagonal are non-zero. Thus, by Lemma 2, we know that $\boldsymbol{A}^*$ has rank $n$ if and only if $det(\boldsymbol{A}^*) \neq 0$. Finally, by Lemma 3, $det(\boldsymbol{A}) \neq 0$ if and only if $det(\boldsymbol{A}^*) \neq 0$. We thus complete the proof. □

## Appendix: Proof of Lemma 3

The claims on Operations 1 and 2 are easy to prove; we leave the proofs to you as exercises.

To prove the claim on Operation 3, we will leverage Corollary 4 (which holds as long as the claim on Operation 1 is true).

Suppose that, after performing Operation 3 on $\boldsymbol{A}$, we obtain a matrix $\boldsymbol{A'}$. Our goal is to show that $det(\boldsymbol{A}) = det(\boldsymbol{A'})$. Let us define a new matrix $\boldsymbol{B}$:

- $\boldsymbol{B}$ is the same as $\boldsymbol{A}$, except that the $i$-th row of $\boldsymbol{B}$ is replaced by the $j$-th row of $\boldsymbol{A}$.

In other words, the $i$-th row of $\boldsymbol{B}$ is identical to the $j$-th row of $\boldsymbol{B}$. Corollary 4 tells us that $det(\boldsymbol{B}) = 0$. Next, we will focus on showing:

$$det(\boldsymbol{A'}) = det(\boldsymbol{A}) + det(\boldsymbol{B}) \quad (3)$$

which will indicate $det(\boldsymbol{A}) = det(\boldsymbol{A'})$ and hence will complete the proof.

Define $a'_{ik}$ as the number at the $i$-th row and $j$-th column of $\boldsymbol{A'}$, and define $a_{ik}$, $b_{ik}$ similarly with respect to $\boldsymbol{A}$, $\boldsymbol{B}$, respectively. Note that:

$$a'_{ik} = a_{ik} + b_{ik}$$

holds by the way $\boldsymbol{A'}$ and $\boldsymbol{B}$ were obtained.

In fact, (3) follows almost directly from the definition of determinants. Let us calculate $det(\boldsymbol{A'})$ by expanding the matrix on row $i$:

$$\begin{aligned}
det(\boldsymbol{A'}) &= \sum_{k=1}^{n}(-1)^{i+k}a'_{ik}\cdot det(\boldsymbol{A'}_{ik}) \\
&= \sum_{k=1}^{n}(-1)^{i+k}(a_{ik}+b_{ik})\cdot det(\boldsymbol{A'}_{ik}) \\
&= \left(\sum_{k=1}^{n}(-1)^{i+k}a_{ik}\cdot det(\boldsymbol{A'}_{ik})\right)+\left(\sum_{k=1}^{n}(-1)^{i+k}b_{ik}\cdot det(\boldsymbol{A'}_{ik})\right) \\
&= \left(\sum_{k=1}^{n}(-1)^{i+k}a_{ik}\cdot det(\boldsymbol{A}_{ik})\right)+\left(\sum_{k=1}^{n}(-1)^{i+k}b_{ik}\cdot det(\boldsymbol{B}_{ik})\right) \\
&= det(\boldsymbol{A})+det(\boldsymbol{B}).
\end{aligned}$$