Exercises: Linear Systems and Matrix Inverse

Problem 1. Consider the following linear system:

$$\left\{\begin{array}{ccc} x_1 + x_2 + x_3 + x_4 & = & 1 \\ 3x_1 + x_2 + x_3 + x_4 & = & a \\ x_2 + 2x_3 + 2x_4 & = & 3 \\ 5x_1 + 4x_2 + 3x_3 + 3x_4 & = & a \end{array}\right.$$

Depending on the value of a, when does the system have no solution, a unique solution, and infinitely many solutions?

Problem 2. Consider the following linear system:

$$\left\{\begin{array}{ccc} 2x_1 + x_2 + bx_3 & = & 0 \\ x_1 + x_2 + bx_3 & = & 0 \\ bx_1 + x_2 + 2x_3 & = & 0 \end{array}\right.$$

Depending on the value of b, when does the system have no solution, a unique solution, and infinitely many solutions?

Problem 3. Use Cramer's rule to solve the following linear system:

$$\left\{\begin{array}{ccc} 2x - 4y & = & -24 \\ 5x + 2y & = & 0 \end{array}\right.$$

Problem 4. Compute the inverse of

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Problem 5. Use the "inverse formula" to calculate the inverse of the matrix in Problem 4.

Problem 6. Compute the inverse of

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 1 \\ -2 & -3 & 1 \\ 5 & 9 & 1 \end{bmatrix}$$

Problem 7. Let $\boldsymbol{A}$ be an $n \times n$ matrix. Also, let $\boldsymbol{I}$ be the $n \times n$ identity matrix. Prove: if $\boldsymbol{A}^3 = \boldsymbol{0}$, then

$$(\boldsymbol{I} - \boldsymbol{A})^{-1} = \boldsymbol{I} + \boldsymbol{A} + \boldsymbol{A}^2.$$

Problem 8. Consider:

$$\boldsymbol{A} = \begin{bmatrix} 2 & 1 & b \\ 1 & 1 & b \\ b & 1 & 2 \end{bmatrix}$$

Under what values of b does $\boldsymbol{A}^{-1}$ exist?