CSCI2100 Tutorial 8

CSCI 2100 Teaching Team, Spring 2023

Review on Hash Table

- S = a set of n integers in $[1, U]$
- Main idea: divide S into a number m of disjoint "buckets"
- Set $m = \Theta(n)$
- Guarantees
  - Space consumption: $O(n)$
  - Preprocessing cost: $O(n)$
  - Query cost: $O(1)$ in expectation

Review on Hash Table

- Divide S into a number m of disjoint buckets:
    - Choose a function h from $[1, U]$ to $[1, m]$
    - For each $i \in [1, m]$, create an empty linked list L_i
    - For each $x \in S$:
        - Compute $h(x)$
        - Insert x into $L_{h(x)}$

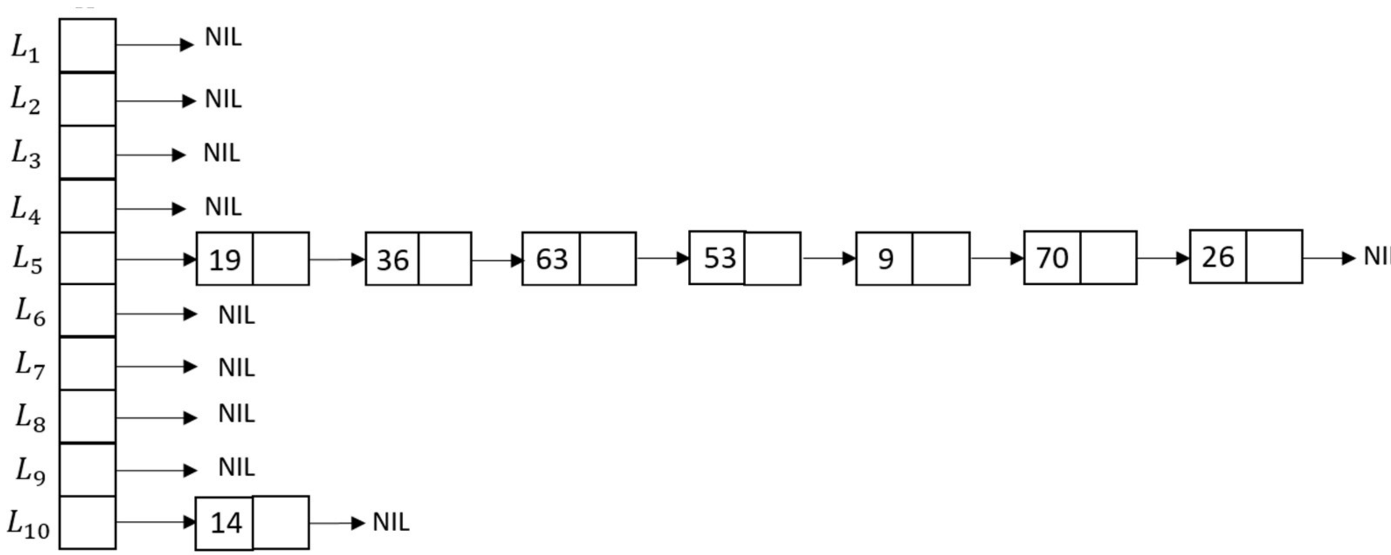
- Important: choose a good hash function h

Review on Hash Table

- Construct a universal family
  - Pick a prime number p such that $p \geq m$ and $p \geq U$
  - Choose an integer α from $[1, p-1]$ uniformly at random
  - Choose an integer β from $[0, p-1]$ uniformly at random
  - Define a hash function:
    $$h(k) = 1 + \big((\alpha k + \beta) \bmod p\big) \bmod m$$

Example

- Let $S = \{19,36,63,53,14,9,70,26\}$
- We choose $m = 10, p = 71$, suppose that α and β are randomly chosen to be 3 and 7, respectively
- $h(k) = 1 + (((3k + 7) \text{mod } 71) \text{ mod } 10)$



Relationships between Hash Functions and Queries

- Let H be the universal family defined in the previous slides
- Given a function $h \in H$ and an integer q $\in [1, U]$:
  - Define $\text{cost}(h, q) = |\{x \in S \mid h(x) = h(q)\}|$

query value

	1	2	…	U
h_1	$\text{cost}(h_1, 1)$	$\text{cost}(h_1, 2)$	…	$\text{cost}(h_1, U)$
h_2	$\text{cost}(h_2, 1)$	$\text{cost}(h_2, 2)$	…	$\text{cost}(h_2, U)$
…	…	….	…	….
$h_{\lvert H \rvert}$	$\text{cost}(h_{\lvert H \rvert}, 1)$	$\text{cost}(h_{\lvert H \rvert}, 2)$	…	$\text{cost}(h_{\lvert H \rvert}, U)$
Average	$O(1)$	$O(1)$	$O(1)$	$O(1)$

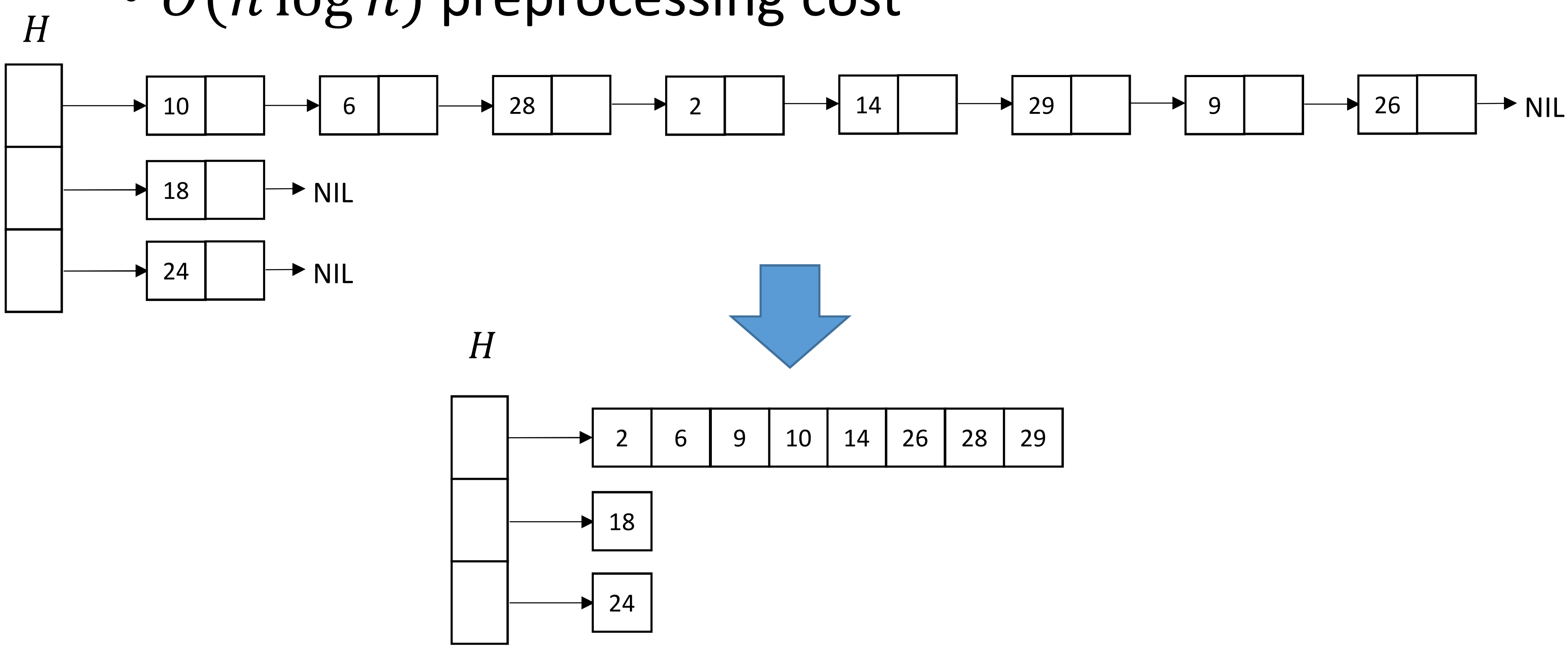
Hash Table

- Worst-case expected query cost: $O(1)$
- Worst-case query cost: $O(n)$

- Question:
  - Can we improve the worst-case query cost?

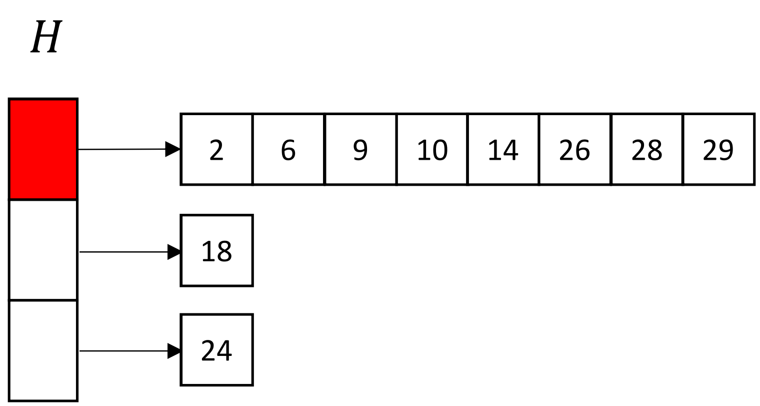
Hash Table: Improving the Worst Cost

- Replace linked lists with sorted arrays
- $O(n \log n)$ preprocessing cost



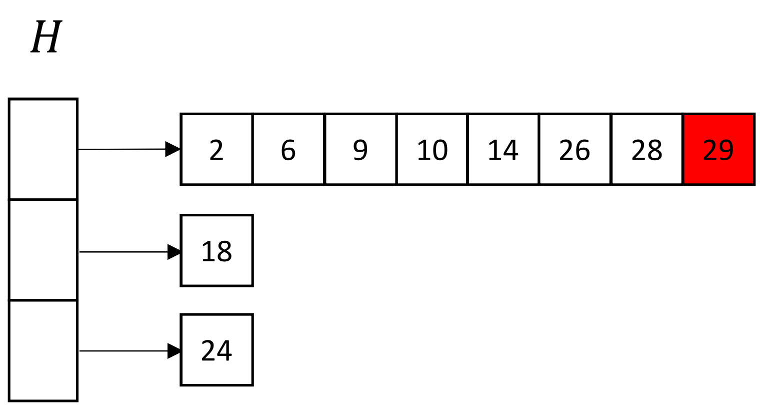
Hash Table: Improving the Worst Cost

- Query: whether 29 exists
- Step 1:
  - Access the hash table to obtain the address of corresponding array
    - $O(1)$ time



Hash Table: Improving the Worst Cost

- Query: whether 29 exists
- Step 2:
  - Perform binary search on the array to find the target
    - $O(\log n)$ time
- Overall worst-case complexity: $O(\log n)$



Hash Table: Improving the Worst Cost

- This method retains the $O(1)$ worst-case expected query time.
- Proof:
  - Suppose we look up an integer q
  - Define random variable $X_{h(q)}$ to be the length of array that corresponds to the hash value $h(q)$
  - Expected query time:

$$\begin{aligned} \mathrm{E}\left[\log_2 X_{h(q)}\right] &= \sum_{l=1}^{n} \log_2 l \Pr\left(X_{h(q)} = l\right) \\ &\le \sum_{l=1}^{n} l \Pr\left(X_{h(q)} = l\right) \\ &= \mathrm{E}\left[X_{h(q)}\right] \\ &= O(1) \end{aligned}$$

The Two-Sum Problem (Revisited)

- Problem Input:
  - An array A of n distinct integers (not necessarily sorted).
- Goal:
  - Determine whether if there exist two different integers x and y in A satisfying $x + y = v$
- Example: find a pair whose sum is 20

11	3	17	7	2	13

Solution 1: Binary Search the Answer

- Goal: Find a pair (x, y) such that $x + y = v$
- Observe that given x, $y = v - x$, is determined
- Solution:
  - Sort A
  - For each x in A:
    - set y as $v - x$
    - Use binary search to see if y exists in the sequence
- Time complexity: $O(n \log n)$

Solution 2: Using the Hash Table

- Step 1 and 2:
  - Choose a hash function h and create an empty hash table H
  - Insert each x in *A* into $L_{h(x)}$

- Step 3:
  - For *i* = 1 to *n*
    - Set y as $v - A[i]$
    - Check if y is in the hash table; if it is, return yes
  - Return no

Time Complexity

- Step 1 and 2: $O(n)$
- Step 3:
  - The step issues *n* queries (one for each *y*)
  - Let X_i be the time of the i-th query
  - We know $E[X_i] = O(1)$
  - The worst-case expected cost of step 3 is $\sum_i E[X_i] = \mathrm{O}(n)$
- Overall: $O(n)$ in expectation

Sorting by Frequency
(a Regular Exercise)

- Problem input:
  - Let S be a multi-set of n integers. The frequency of an integer x as the number of occurrences of x in S.

- Goal: Produce an array that sorts the distinct integers in S by frequency.

input:

10	8	8	12	9	9	12	12

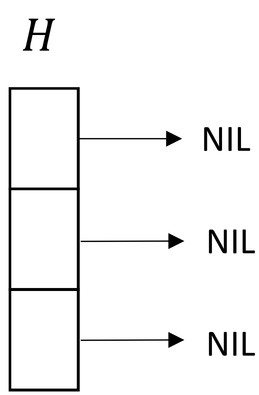
output:

12	8	9	10

12 : 3 occurrences
8 : 2 occurrences
9 : 2 occurrences
10 : 1 occurrence

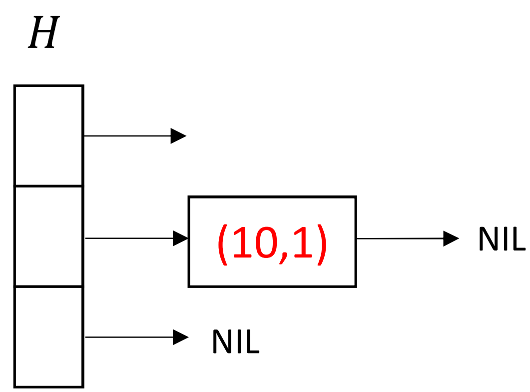
Using a Hash Table to Obtain Frequencies

10	8	8	12	9	9	12	12



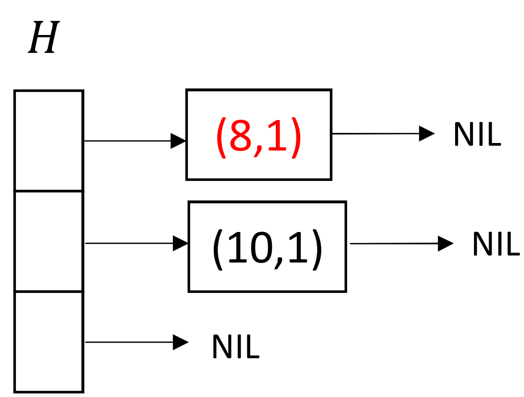
Using a Hash Table to Obtain Frequencies

10	8	8	12	9	9	12	12



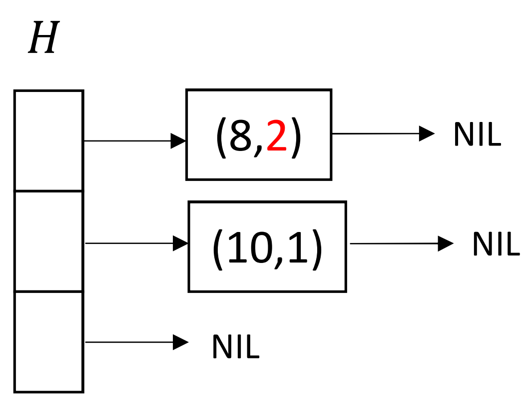
Using a Hash Table to Obtain Frequencies

10	8	8	12	9	9	12	12



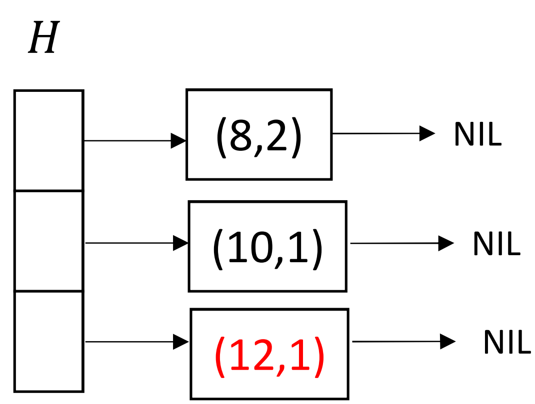
Using a Hash Table to Obtain Frequencies

10	8	8	12	9	9	12	12



Using a Hash Table to Obtain Frequencies

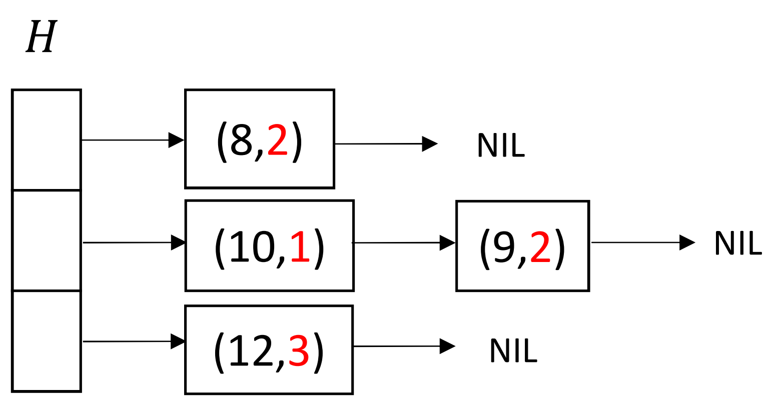
10	8	8	12	9	9	12	12



Using a Hash Table to Obtain Frequencies

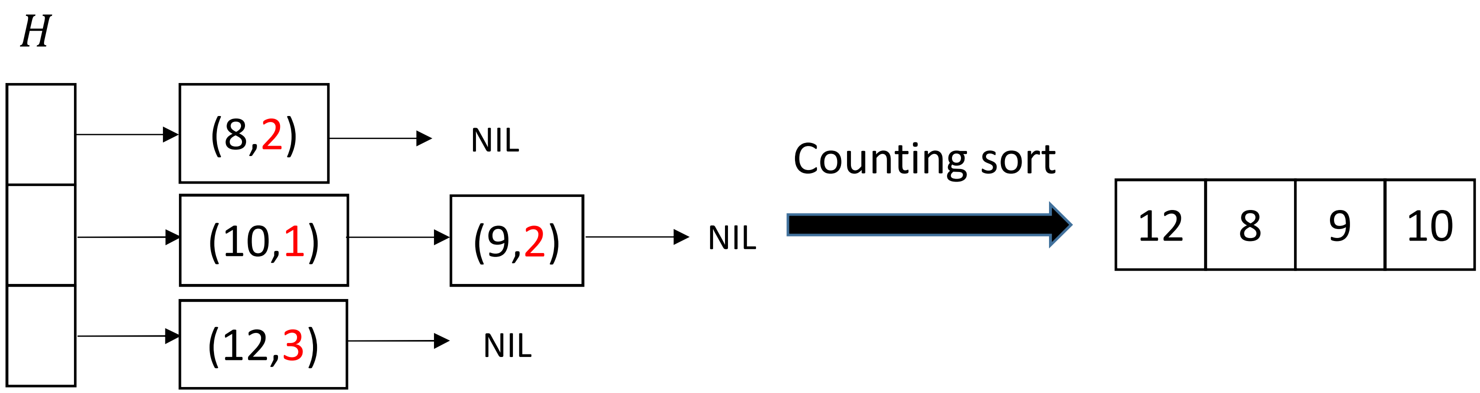
- The final state:

10	8	8	12	9	9	12	12



Counting Sort!

- Now we sort the numbers by frequency.
- Key observation: each frequency is in $[1, n]$.
- We can carry out the sorting with counting sort in $O(n)$ time.



Total time complexity: $O(n)$ expected time.