Single Source Shortest Paths with Positive Weights

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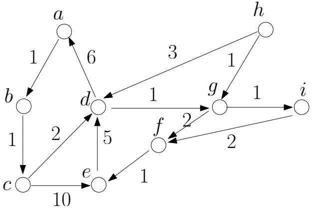
In this lecture, we will revisit the **single source shortest path** (SSSP) problem. Recall that we have already learned that BFS solves the problem efficiently when all the edges have the **same** weight. Today, we will see how to solve the problem in a more general situation where the edges can have arbitrary positive weights.

Weighted Graphs

Let $G = (V, E)$ be a directed graph. Let w be a function that maps each edge in E to a positive integer value. Specifically, for each $e \in E$, $w(e)$ is a **positive** integer value, which we call the **weight** of e.

A **directed weighted graph** is defined as the pair (G, w).

Example



The integer on each edge indicates its weight. For example, $w(d, g) = 1$, $w(g, f) = 2$, and $w(c, e) = 10$.

Shortest Path

Consider a directed weighted graph defined by a directed graph $G = (V, E)$ and function w.

Consider a path in G: $(v_1, v_2), (v_2, v_3), ..., (v_\ell, v_{\ell+1})$, for some integer $\ell \geq 1$. We define the **length** of the path as

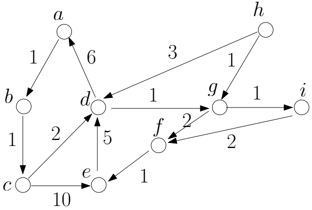
$$\sum_{i=1}^{\ell} w(v_i, v_{i+1}).$$

Recall that we may also denote the path as $v_1 \to v_2 \to ... \to v_{\ell+1}$.

Given two vertices $u, v \in V$, a **shortest path** from u to v is a path from u to v that has the minimum length among all the paths from u to v.

If v is unreachable from u, then the shortest path distance from u to v is ∞.

Example



- The path $c \to e$ has length 10.
- The path $c \to d \to g \to f \to e$ has length 6.

The first path is a shortest path from c to e.

Single Source Shortest Path (SSSP) with Positive Weights

Let (G, w) with $G = (V, E)$ be a directed weighted graph, where w maps every edge of E to a positive value.

Given a vertex s in V, the goal of the **SSSP problem** is to find, for every other vertex $t \in V \setminus \{s\}$, a shortest path from s to t, unless t is unreachable from s.

Next, we will first explain the Dijkstra's algorithm for solving the SSSP problem, which outputs a **shortest path tree** that encodes all the shortest paths from the source vertex s.

The Edge Relaxation Idea

For every vertex $v \in V$, we will — at all times — maintain a value $dist(v)$ that represents the length of the shortest path from s to v **found so far**.

At the end of the algorithm, we will ensure that every $dist(v)$ equals the precise shortest path distance from s to v.

A core operation in our algorithm is called **edge relaxation**:

- Given an edge (u, v), we **relax** it as follows:
  - If $dist(v) < dist(u) + w(u, v)$, do nothing;
  - Otherwise, reduce $dist(v)$ to $dist(u) + w(u, v)$.

Dijkstra's Algorithm

1. Set $parent(v) = \text{nil}$ for all vertices $v \in V$
2. Set $dist(s) = 0$, and $dist(v) = \infty$ for all other vertices $v \in V$
3. Set $S = V$
4. Repeat the following until S is empty:

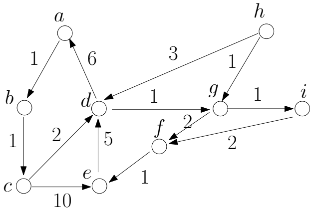
   5.1 Remove from S the vertex u with the smallest $dist(u)$.
   /* next we relax all the outgoing edges of u */

   5.2 for every outgoing edge (u, v) of u

      5.2.1 if $dist(v) > dist(u) + w(u, v)$ then
      set $dist(v) = dist(u) + w(u, v)$, and $parent(v) = u$

Example

Suppose that the source vertex is c.

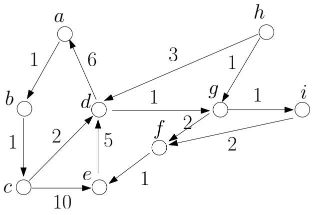


vertex v	$dist(v)$	$parent(v)$
a	∞	nil
b	∞	nil
c	0	nil
d	∞	nil
e	∞	nil
f	∞	nil
g	∞	nil
h	∞	nil
i	∞	nil

$S = \{a, b, c, d, e, f, g, h, i\}$.

Example

Relax the out-going edges of c (because $dist(c)$ is the smallest in S):

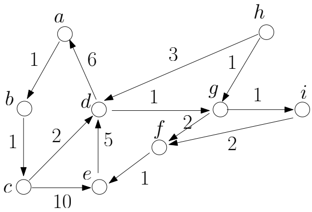


vertex v	$dist(v)$	$parent(v)$
a	∞	nil
b	∞	nil
c	0	nil
d	2	c
e	10	c
f	∞	nil
g	∞	nil
h	∞	nil
i	∞	nil

$S = \{a, b, d, e, f, g, h, i\}$.
Note that c has been removed!

Example

Relax the out-going edges of d (because $dist(d)$ is the smallest in S):

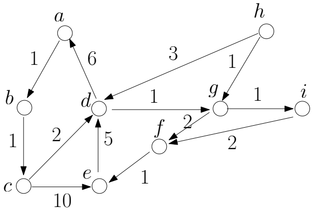


vertex v	$dist(v)$	$parent(v)$
a	8	d
b	∞	nil
c	0	nil
d	2	c
e	10	c
f	∞	nil
g	3	d
h	∞	nil
i	∞	nil

$S = \{a, b, e, f, g, h, i\}$.

Example

Relax the out-going edges of g:

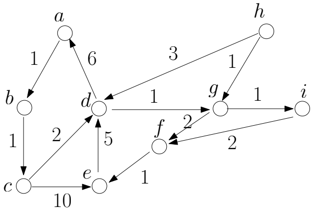


vertex v	$dist(v)$	$parent(v)$
a	8	d
b	∞	nil
c	0	nil
d	2	c
e	10	c
f	5	g
g	3	d
h	∞	nil
i	4	g

$S = \{a, b, e, f, h, i\}$.

Example

Relax the out-going edges of i:

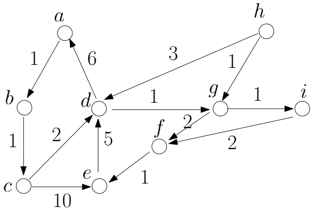


vertex v	$dist(v)$	$parent(v)$
a	8	d
b	∞	nil
c	0	nil
d	2	c
e	10	c
f	5	g
g	3	d
h	∞	nil
i	4	g

$S = \{a, b, e, f, h\}$.

Example

Relax the out-going edges of f:

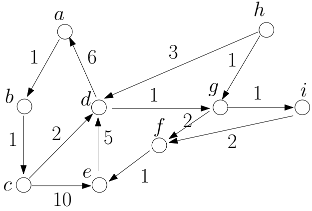


vertex v	$dist(v)$	$parent(v)$
a	8	d
b	∞	nil
c	0	nil
d	2	c
e	6	f
f	5	g
g	3	d
h	∞	nil
i	4	g

$S = \{a, b, e, h\}$.

Example

Relax the out-going edges of e:

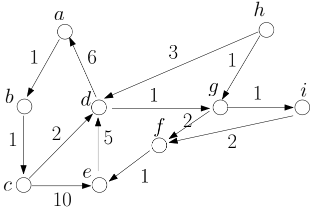


vertex v	$dist(v)$	$parent(v)$
a	8	d
b	∞	nil
c	0	nil
d	2	c
e	6	f
f	5	g
g	3	d
h	∞	nil
i	4	g

$S = \{a, b, h\}$.

Example

Relax the out-going edges of a:

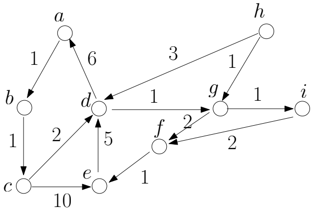


vertex v	$dist(v)$	$parent(v)$
a	8	d
b	9	a
c	0	nil
d	2	c
e	6	f
f	5	g
g	3	d
h	∞	nil
i	4	g

$S = \{b, h\}$.

Example

Relax the out-going edges of b:

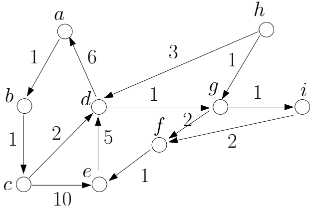


vertex v	$dist(v)$	$parent(v)$
a	8	d
b	9	a
c	0	nil
d	2	c
e	6	f
f	5	g
g	3	d
h	∞	nil
i	4	g

$S = \{h\}$.

Example

Relax the out-going edges of h:



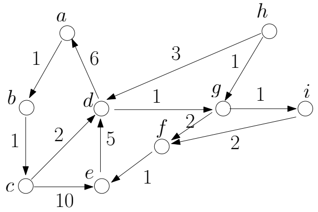
vertex v	$dist(v)$	$parent(v)$
a	8	d
b	9	a
c	0	nil
d	2	c
e	6	f
f	5	g
g	3	d
h	∞	nil
i	4	g

$S = \{\}$.
All the shortest path distances are now final.

Constructing the Shortest Path Tree

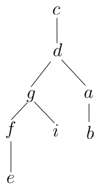
For every vertex v, if $u = parent(v)$ is not nil, then make v a child of u.

Example



vertex v	$parent(v)$
a	d
b	a
c	nil
d	c
e	f
f	g
g	d
h	nil
i	g

shortest path tree



Running Time

It will be left as an exercise for you to to implement Dijkstra's algorithm in $O(|V| + |E| \cdot \log |V|)$ time. You have already learned all the data structures for this purpose. Now it is time to practice using them.

Correctness

Lemma: When vertex v is removed from S, $dist(v)$ equals precisely the shortest path distance — denoted as $spdist(v)$ — from s to v.

The correctness of Dijkstra's algorithm follows from the lemma.

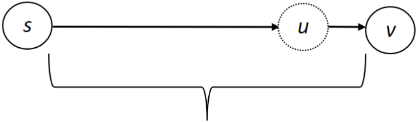
Correctness

We will prove the claim by induction on the sequence of vertices removed.

- Base case:
  This is obviously true for the first vertex removed, which is s itself with $dist(s) = 0$.
- Inductive:
  Assume the claim is true with respect to all the vertices already removed. Let v be the next node to be removed. We need to prove $dist(v) = spdist(v)$.

Correctness

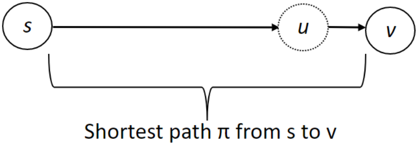
Consider an arbitrary shortest path π from s to v. Let u be the vertex right before v on π.



Claim: u must have been removed from S.

Our target lemma follows from the above claim because, by our inductive assumption, $dist(u) = spdist(u)$ when u was removed. Then, the algorithm relaxed the edge (u, v), which must have set $dist(v) = spdist(u) + w(u, v) = spdist(v)$.

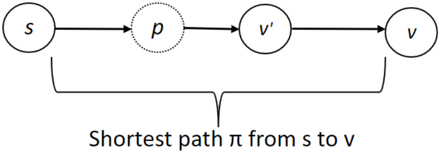
Correctness



Stronger claim: All the nodes on π from s to u must have been removed.

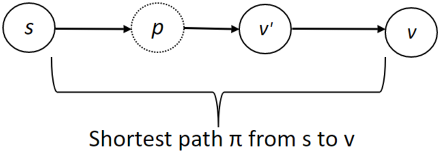
Correctness

We will prove the stronger claim by contradiction.



Suppose the statement is not true. When v is to be removed from S, another vertex on π — let it be v' — still remains in S. Define p as the vertex right before v' on π.

Correctness



By the inductive assumption, $dist(p) = spdist(p)$ when p was removed. Hence, after relaxing the edge (p, v'), we have $dist(v') = spdist(p) + w(p, v') = spdist(v')$.

But this means $dist(v') = spdist(v') < spdist(v) \leq dist(v)$!

Hence, v' should be the next vertex to be removed from S, contradicting the definition of v. $\square$