Graphs and Trees: Basic Concepts and Properties (Discrete Math Review)

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This lecture formally defines **graphs** and **trees**, and proves some of their basic properties.

Undirected Graphs

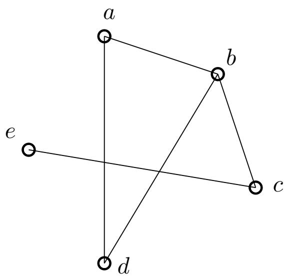
An **undirected simple graph** is a pair of (V, E) where:

- V is a set of elements;
- E is a set of **unordered pairs** $\{u, v\}$ such that u and v are **distinct** elements in V.

Each element in V is called a **node** or a **vertex**. Each pair in E is called an **edge**.

An edge $\{u, v\}$ is said to be **incident** to vertices u and v; the two vertices are said to be **adjacent** to each other.

Example



This is a graph (V, E) where

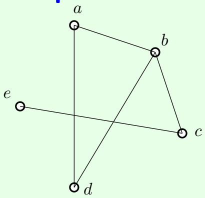
- $V = \{a, b, c, d, e\}$
- $E = \{\{a, b\}, \{b, c\}, \{a, d\}, \{b, d\}, \{c, e\}\}$.
    - The number of edges equals $|E| = 5$.

Vertex-Induced Graphs

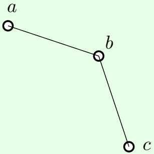
Let $G = (V, E)$ be an undirected graph. Fix a subset $V' \subseteq V$. The subgraph of G **induced by** V' is (V', E') where

$$E' = \{\{u, v\} \in E \mid u \in V' \text{ and } v \in V'\}.$$

Example:



G



subgraph of G induced by $\{a, b, c\}$

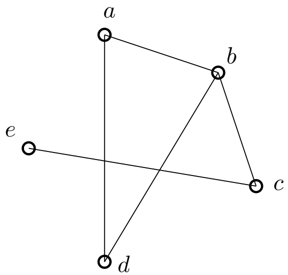
Paths and Cycles

Let $G = (V, E)$ be an undirected simple graph. A **path** in G is a sequence of nodes $(v_1, v_2, ..., v_k)$ such that

- v_i and v_{i+1} are adjacent, for each $i \in [1, k-1]$.

A **cycle** in G is a path $(v_1, v_2, ..., v_k)$ such that $k \geq 4$ and $v_1 = v_k$.

Example

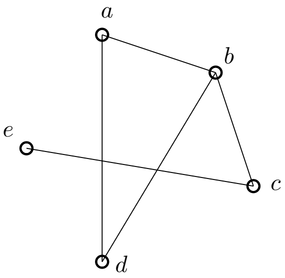


(a, b, d, a) is a cycle, whereas (a, b, c, e) is a path but not a cycle.

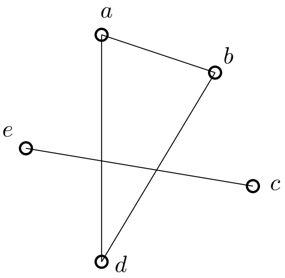
Connected Graphs

An undirected graph $G = (V, E)$ is **connected** if, for any two distinct vertices u and v, G has a path from u to v.

Example



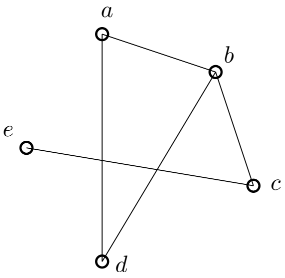
connected



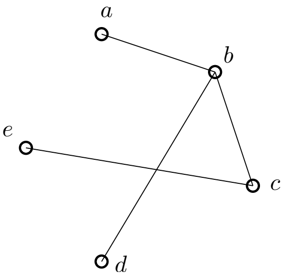
not connected

Trees

A **tree** is a connected undirected graph with no cycles.



not a tree



a tree

A Property

Lemma: A tree with n nodes has $n - 1$ edges.

The proof will be left to you as an exercise.

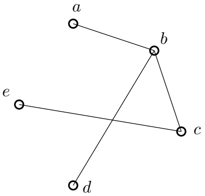
Rooting a Tree

Given any tree T and an arbitrary node r, we can allocate a **level** to each node as follows:

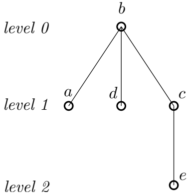
- r is the **root** of T — this is **level 0** of the tree.
- All the nodes that are 1 edge away from r constitute **level 1** of T.
- All the nodes that are 2 edges away from r constitute **level 2** of T.
- And so on.

The number of levels is called the **height** of T. We say that T has been **rooted** once a root has been designated.

Example

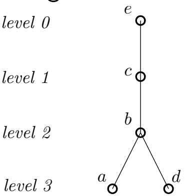


Rooting the tree at *b*



Height 3

Rooting the tree at *e*



Height 4

Concepts on Rooted Trees — Parents and Children

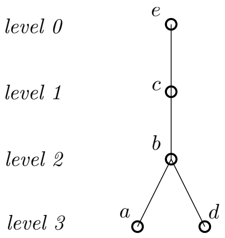
Consider a tree T that has been rooted.

Let u and v be two nodes in T. We say that u is the **parent** of v if

- the level of v is one more than that of u, and
- u and v are adjacent.

Accordingly, we say that v is a **child** of u.

Example



Node b is the parent of two child nodes: a, d.
Node e is the parent of c, which is in turn the parent of b.

Concepts on Rooted Trees — Ancestors and Descendants

Consider a rooted tree T.

Let u and v be two nodes in T. We say that u is an **ancestor** of v if one of the following holds:

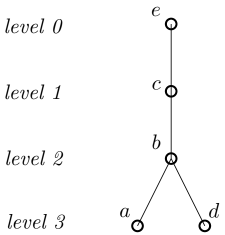
- the level of u is at most that of v;
- u has a path to v.

Note: A node is an ancestor of itself.

Accordingly, if u is an ancestor of v, then v is a **descendant** of u.

In particular, if $u \neq v$, we say that u is a **proper ancestor** of v, and likewise, v is a **proper descendant** of u.

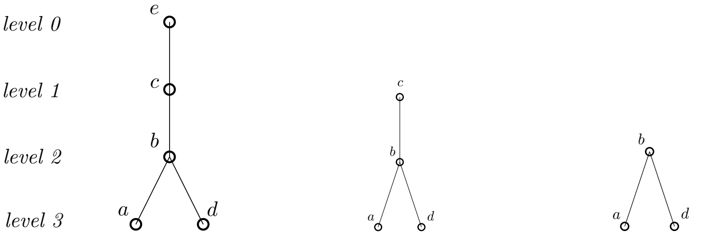
Example



Node b is an ancestor of b, a and d.
Node c is an ancestor of c, b, a, and d.
Node c is a proper ancestor of b, a, d.

Concepts on Rooted Trees — Subtrees

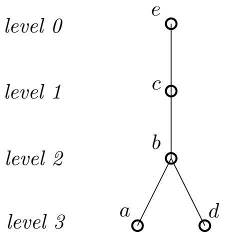
Let u be a node in a rooted tree T. Let T_u be the subgrpah of T induced by the set of descendants of u. The **subtree** of u is the rooted tree obtained by rooting T_u at u.



tree — subtree of c — subtree of b

Concepts on Rooted Trees — Internal and Leaf Nodes

In a rooted tree, a node is a **leaf** if it has no children; otherwise, it is an **internal node**.



Internal nodes: e, c, and b. Leaf nodes: a and d.

A Property

Lemma: Let T be a rooted tree where every internal node has at least 2 child nodes. If m is the number of leaf nodes, then the number of internal nodes is at most $m - 1$.

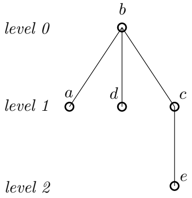
Proof: Consider the tree as the schedule of a tournament described as follows. The competing teams are initially placed at the leaf nodes. Each internal node v represents a match among the teams at the child nodes, such that only the winning team advances to v. The team winning the match at the root is the champion.

Each match eliminates at least one team. There are at most $m - 1$ teams to eliminate before the champion is determined. Hence, there can be at most $m - 1$ matches (i.e., nodes). □

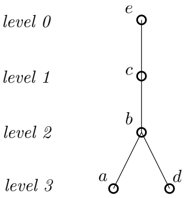
Concepts on Rooted Trees — k-Ary and Binary

A ***k*-ary tree** is a rooted tree where every internal node has at most k child nodes.

A 2-ary tree is called a **binary tree**.



3-ary



Binary

Concepts on a Binary Tree—Left and Right

A binary tree is **left-right labeled** if

- Every node v — except the root — has been designated either as a **left** or **right** node of its parent.
- Every internal node has at most one left child, and at most one right child.

> Throughout this course, we will discuss only binary trees that have been left-right labeled. Because of this, by a “binary tree”, we always refer to a left-right labeled one.

Concepts on a Binary Tree — Left and Right

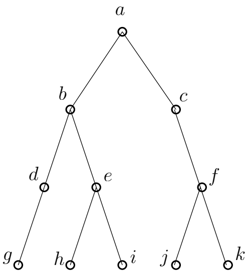
A (left-right labeled) binary tree implies an ordering among the nodes at the same level.

Let u and v be nodes at the same level with parents p_u and p_v, respectively. We say that u is on the left of v if either of the following holds:

- $p_u = p_v$ and u is the left child (implying that v is the right child);
- $p_u \neq p_v$ and p_u is on the left of p_v.

Accordingly, we say that v is on the right of u.

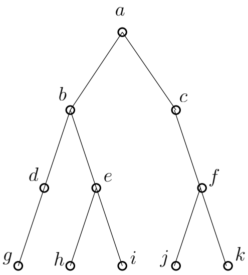
Example



At Level 1, b is on the left of c.
At Level 2, the nodes from left to right are d, e, and f.
At Level 3, the nodes from left to right are g, h, i, j, and k.

Concepts on a Binary Tree — Full Level

Consider a binary tree with height h. Its level ℓ $(0 \leq \ell \leq h-1)$ is **full** if it contains 2^ℓ nodes.



Levels 0 and 1 are full, but levels 2 and 3 are not.

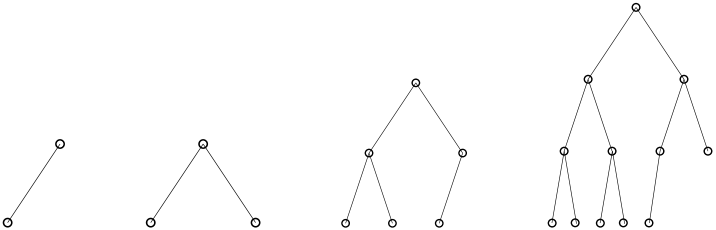
Concepts on a Binary Tree — Complete Binary Tree

A binary tree of height h is **complete** if:

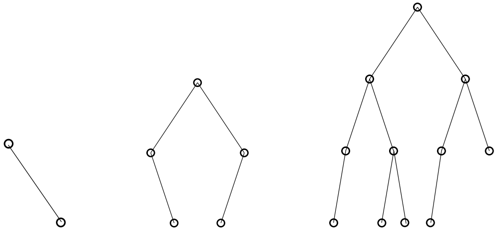
- Levels 0, 1, ..., $h-2$ are all full (i.e., the only possible exception is the bottom level).
- At level $h-1$, the leaf nodes are as far left as possible.

Example

Complete binary trees:



Not complete binary trees:



A Property

Lemma: A complete binary tree with $n \geq 2$ nodes has height $O(\log n)$.

Proof: Let h be the height of the binary tree. As Levels $0, 1, ..., h-2$ are full, we know that

$$\begin{aligned} 2^0 + 2^1 + ... + 2^{h-2} &\leq n \\ \Rightarrow \quad 2^{h-1} - 1 &\leq n \\ \Rightarrow \quad h &\leq 1 + \log_2(n+1) = O(\log n). \end{aligned}$$

□