Hashing

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This lecture will revisit the **dictionary search** problem, where we want to locate an integer q in a set of size n or declare the absence of q. Binary search solves the problem in $O(\log n)$ time (assuming a sorted array on the n integers). We will reduce the cost to $O(1)$ **in expectation** with a structure called the **hash table**.

The Dictionary Search Problem (Redefined)

S is a set of n integers. We want to preprocess S into a data structure to answer the following queries efficiently:

- **(Dictionary search) query:** given an integer q, decide whether $q \in S$.

We will measure a data structure's performance by:

- Space consumption: the number of memory cells occupied;
- Query cost: query time;
- Preprocessing cost: time of building the structure.

Dictionary Search — Solution Based on Binary Search

We can solve the problem by storing S in a sorted array of length n and answering a query with binary search. This ensures:

- Space consumption: $O(n)$;
- Query cost: $O(\log n)$;
- Preprocessing cost: $O(n \log n)$.

Dictionary Search — This Lecture (Hash Table)

We will improve the previous solution in expectation:

- Space consumption: $O(n)$
- Query cost: $O(\log n) \Rightarrow O(1)$ in expectation;
- Preprocessing cost: $O(n \log n) \Rightarrow O(n)$.

Hashing

Main idea: divide S into small **disjoint** subsets such that a query only needs to search **one** subset.

We assume that every integer is in $[1, U]$.
Denote by $[m]$ the set of integers from 1 to m.

> A **hash function** h is a function from $[U]$ to $[m]$. Namely, given any integer k, the function's output $h(k)$ is an integer in $[m]$.
>
> The value $h(k)$ is called the **hash value** of k.

Hash Table — Preprocessing

First, choose an integer $m > 0$, and a hash function h from $[U]$ to $[m]$.

Then, preprocess S as follows:

1. Create an array H of length m.
2. For each $i \in [1, m]$, create an empty linked list L_i. Keep the head and tail pointers of L_i in $H[i]$.
3. For each integer $x \in S$:
    - Calculate the hash value $h(x)$.
    - Insert x into $L_{h(x)}$.

Space consumption: $O(n + m)$.
Preprocessing time: $O(n + m)$.

Hash Table — Querying

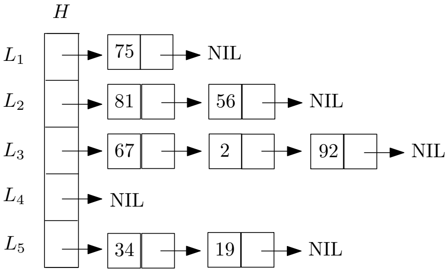
We answer a query with value q as follows:

1. Calculate the hash value $h(q)$.
2. Scan the whole $L_{h(q)}$. If q is not found, answer "no"; otherwise, answer "yes".

Query time: $O(|L_{h(v)}|)$, where $|L_{h(v)}|$ is the number of elements in $L_{h(v)}$.

Example

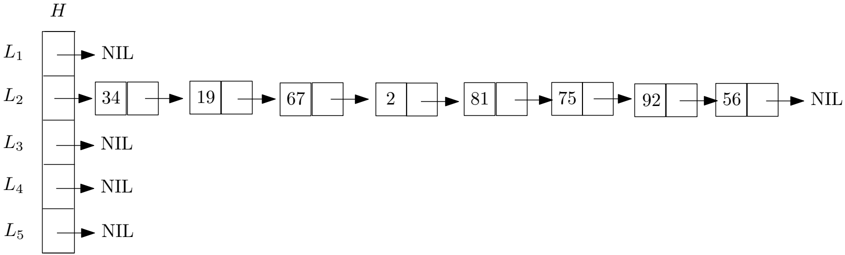
Let $S = \{34, 19, 67, 2, 81, 75, 92, 56\}$. Suppose that we choose $m = 5$ and $h(k) = 1 + (k \mod m)$.



To answer a query with $q = 57$, we scan all the elements in L_3 and answer "no". For this hash function, the maximum query time is the cost of scanning a linked list of 3 elements.

Example

Let $S = \{34, 19, 67, 2, 81, 75, 92, 56\}$. Suppose that we choose $m = 5$, and $h(k) = 2$.



For this hash function, the maximum query time is the cost of scanning a linked list of 8 elements (i.e., the worst possible).

A good hash function should create linked lists of roughly the same size.

Next we will introduce a technique that can choose a good hash function to guarantee $O(1)$ expected query time.

Let $\mathcal{H}$ be a family of hash functions from $[U]$ to $[m]$. $\mathcal{H}$ is **universal** if the following holds:

Let k_1, k_2 be two distinct integers in $[U]$. By picking a function $h \in \mathcal{H}$ uniformly at random, we guarantee that

$$\boldsymbol{Pr}[h(k_1) = h(k_2)] \quad \leq \quad 1/m.$$

We will prove that universality ensures $O(1)$ expected query time. Then, we will describe a way to obtain such a good hash function.

Analysis of Query Time under Universality

We focus on the case where q does not exist in S (the case where it does is similar). Recall that our algorithm probes all the elements in the linked list $L_{h(q)}$. The query cost is therefore $O(|L_{h(q)}|)$.

Define random variable X_i ($i \in [1, n]$) to be 1 if the i-th element e of S has the same hash value as q (i.e., $h(e) = h(q)$), and 0 otherwise. Thus:

$$|L_{h(q)}| = \sum_{i=1}^{n} X_i$$

Analysis of Query Time under Universality

By universality, $\textbf{\textit{Pr}}[X_i = 1] \le 1/m$, meaning that

$$\begin{aligned} \textbf{\textit{E}}[X_i] &= 1 \cdot \textbf{\textit{Pr}}[X_i = 1] + 0 \cdot \textbf{\textit{Pr}}[X_i = 0] \\ &\le 1/m. \end{aligned}$$

Hence:

$$\textbf{\textit{E}}[|L_{h(q)}|] = \sum_{i=1}^{n} \textbf{\textit{E}}[X_i] \le n/m.$$

By choosing $m = \Theta(n)$, we have $n/m = O(1)$.

Designing a Universal Function

We now construct a universal family $\mathcal{H}$ of hash functions from $[U]$ to $[m]$.

- Pick a prime number p such that $p \geq m$ and $p \geq U$.
- For every $\alpha \in \{1, 2, ..., p-1\}$ and every $\beta \in \{0, 1, ..., p-1\}$, define:

$$h_{\alpha,\beta}(k) = 1 + (((\alpha k + \beta) \mod p) \mod m).$$

- This defines $p(p-1)$ hash functions, which constitute our $\mathcal{H}$.

The proof of universality can be found in the appendix (not required for CSCI2100)

Existence of the Prime Number

Is it always possible to choose a desired prime number p?

Recall that the RAM model is defined with a word length w, namely, the number of bits in a word. Hence, $U \leq 2^w - 1$.

Number theory shows that there is at least one prime number between x and $2x$. Hence, one can prepare in advance such a prime number p in the range $[2^w, 2^{w+1}]$ and use this p to construct a universal hash family.

We have shown that, for any set S of n integers, it is always possible to construct a hash table with the following guarantees on the dictionary search problem:

- Space $O(n)$.
- Preprocessing time $O(n)$.
- Query time $O(1)$ **in expectation**.

Appendix: Proof of Universality
(not required for CSCI2100)

The Prime Ring

Denote by $\mathbb{Z}_p$ the set of integers $\{0, 1, ..., p-1\}$. $\mathbb{Z}_p$ forms a **commutative ring** under "+" and "·" (**both defined using modulo p**). This means:

- $\mathbb{Z}_p$ is closed under $+$ and $\cdot$.
- $+$ satisfies commutativity and associativity.
  - $a + b = b + a \pmod p$ and $a + b + c = a + (b + c) \pmod p$
- $+$ has a zero element, that is, $0 + a = a \pmod p$.
- Every element a has an **additive inverse** $-a$, that is, $a + (-a) = 0 \pmod p$.
- $\cdot$ satisfies commutativity and associativity.
  - $a \cdot b = b \cdot a \pmod p$ and $a \cdot b \cdot c = a \cdot (b \cdot c) \pmod p$
- $\cdot$ modulo p has a **one element**, that is, $1 \cdot a = a \pmod a$.
- $+$ and $\cdot$ satisfy distributivity.
  - $a \cdot (b + c) = a \cdot b + a \cdot c \pmod p$
  - $(b + c) \cdot a = b \cdot a + c \cdot a \pmod p$

The Prime Ring

The ring $\mathbb{Z}_p$ has several crucial properties. Let us start with:

Lemma: Let a be a non-zero element in $\mathbb{Z}_p$. Then, $a \cdot j \neq a \cdot k \pmod{p}$ for any $j, k \in \mathbb{Z}_p$ with $j \neq k$.

Proof: Suppose without loss of generality $j > k$. Assume $a \cdot j = a \cdot k \pmod{p}$, then $a \cdot (j - k) = 0 \pmod{p}$. This means that $a \cdot (j - k)$ must be a multiple of p. Since p is prime, either a or $j - k$ must be a multiple of p. This is impossible because a and $j - k$ are non-zero elements in $\mathbb{Z}_p$. □

The lemma implies that $a \cdot 0$, $a \cdot 1$, ..., $a \cdot (p - 1)$ must take unique values in $\{0, 1, ..., p - 1\}$.

The Prime Ring

The previous lemma implies:

Corollary: Every non-zero element a has a unique multiplicative inverse a^{-1}, namely, $a \cdot a^{-1} = 1 \pmod{p}$.

In other words, $\mathbb{Z}_p$ is a **division ring**.

The Prime Ring

The next property then follows:

Lemma: Every equation $a \cdot x + b = c \pmod{p}$ where a, b, c are in $\mathbb{Z}_p$ and $a \neq 0$ has a unique solution in $\mathbb{Z}_p$.

Proof:

$$\begin{aligned} a \cdot x &= c - b \pmod{p} \\ \Leftrightarrow \quad x &= a^{-1} \cdot (c - b) \pmod{p} \end{aligned}$$

□

Proof of Universality

Next, we will prove that the hash family $\mathcal{H}$ we constructed in Slide 15 is universal. As before, let k_1 and k_2 be distinct integers in $[U]$.

Fact 1: Let

$$\begin{aligned} g_{\alpha,\beta}(k_1) &= (\alpha \cdot k_1 + \beta) \mod p \\ g_{\alpha,\beta}(k_2) &= (\alpha \cdot k_2 + \beta) \mod p \end{aligned}$$

We must have: $g_{\alpha,\beta}(k_1) \neq g_{\alpha,\beta}(k_2)$.

Proof: Otherwise, it must hold that

$$\begin{aligned} & \alpha \cdot k_1 + \beta = \alpha \cdot k_2 + \beta \pmod p \\ \Rightarrow \quad & \alpha \cdot (k_1 - k_2) = 0 \pmod p \end{aligned}$$

which is not possible. $\square$.

Proof of Universality

How many different choices are there for the pair $(g(k_1), g(k_2))$? The answer is at most $p(p-1)$ according to Fact 1: there are p^2 possible pairs in $\mathbb{Z}_p \times \mathbb{Z}_p$ but we need to exclude the p pairs where the two values are the same.

Recall that $\mathcal{H}$ has $p(p-1)$ functions.

Next, we will prove a one-to-one mapping between the possible choices of $(g(k_1), g(k_2))$ and the hash functions in $\mathcal{H}$.

Proof of Universality

Fact 2: Fix any two $x, y \in \mathbb{Z}_p$ such that $x \neq y$. There is a unique pair (α, β) — with $\alpha \in \{1, 2, ..., p-1\}$ and $\beta \in \{0, 1, ..., p-1\}$ — that makes $g_{\alpha,\beta}(k_1) = x$ and $g_{\alpha,\beta}(k_2) = y$.

Proof: Suppose that h is determined by α, β selected as explained in Slide 15. Thus:

$$\begin{aligned} \alpha \cdot k_1 + \beta &= x \pmod{p} \\ \alpha \cdot k_2 + \beta &= y \pmod{p} \end{aligned}$$

Hence:

$$\begin{aligned} \alpha \cdot (k_1 - k_2) &= x - y \pmod{p} \\ \Rightarrow \quad \alpha &= (k_1 - k_2)^{-1} \cdot (x - y) \pmod{p} \\ \Rightarrow \quad \beta &= x - (k_1 - k_2)^{-1} \cdot (x - y) \cdot k_1 \pmod{p} \end{aligned}$$

□

Proof of Universality

Let P be the set of pairs (x, y) such that $x, y \in \mathbb{Z}_p$ and $x \neq y$.

By choosing α, β randomly in their respective ranges, we set $(g_{\alpha,\beta}(k_1), g_{\alpha,\beta}(k_2))$ to a pair $(x, y) \in P$ chosen uniformly at random.

Notice that $h(k_1) = h(k_2)$ if and only if $g_{\alpha,\beta}(k_1) = g_{\alpha,\beta}(k_2) \pmod m$. So now the question boils down to: how many pairs (x, y) in P satisfy $x = y \pmod m$?

Proof of Universality

How many pairs (x, y) in P satisfy $x = y \pmod{m}$?

- For $x = 0$, y can take $m, 2m, 3m, \ldots$. The number of such y's is no more than $\lceil p/m \rceil - 1 \le (p-1)/m$.
- For $x = 1$, y can take $m+1, 2m+1, 3m+1, \ldots$. The number of such y's is no more than $\lceil p/m \rceil - 1 \le (p-1)/m$.
- ...

Hence, the number of such pairs is no more than $p(p-1)/m = |P|/m$.

Now we conclude that the probability of $h(k_1) = h(k_2)$ is at most $1/m$.