Merge Sort

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In this lecture, we will design the **merge sort** which sorts n elements in $O(n \log n)$ time. The algorithm illustrates a **divide and conquer** technique based on recursion.

Recall:

The Sorting Problem

Problem Input:

A set S of n integers is given in an array of length n. The value of n is inside the CPU.

Goal:

Produce an array to store the integers of S in ascending order.

Recall the principle of recursion:

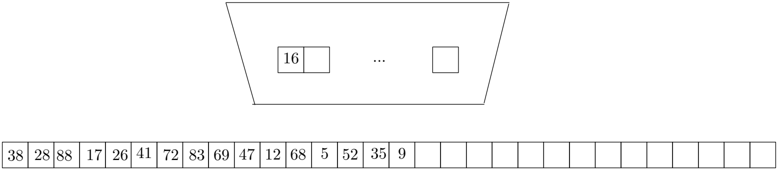
> When dealing with a subproblem (same problem but with a smaller input), consider it solved, and use the subproblem's output to continue the algorithm design.

Merge Sort (Divide and Conquer)

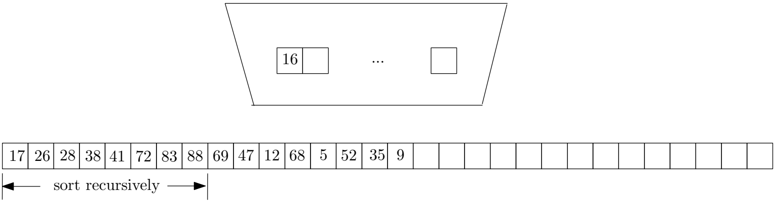
1. Sort the first half of the array S (i.e., a **subproblem** of size $n/2$).
2. Sort the second half of the array S (i.e., a **subproblem** of size $n/2$).
3. Consider both subproblems solved and merge the two halves of the array into the final sorted sequence (details later).

Example

Input:

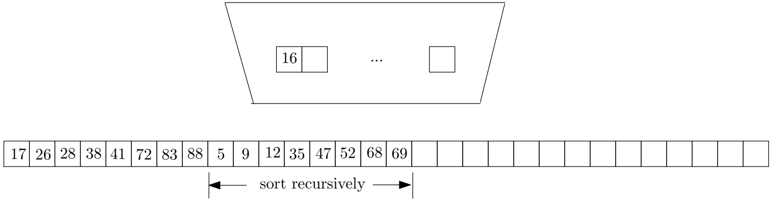


First step, sort the first half of the array by recursion.

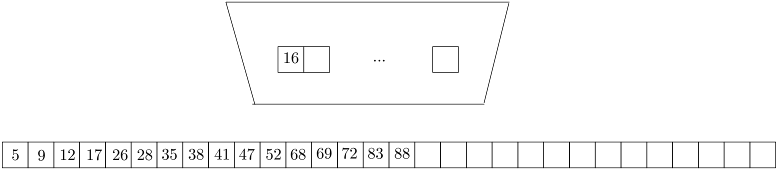


Example

Second step, sort the second half of the array by recursion:



Third step, merge the two halves.

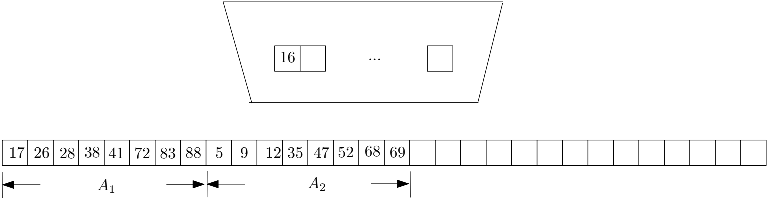


Merging

We are looking at the following **merging problem**.

There are two arrays — denoted as A_1 and A_2 — each containing (at most) $n/2$ integers in ascending order. The goal is to produce a sorted array A containing all the integers in A_1 and A_2.

The following shows an example of the input:



Merging

At the beginning, set $i = j = 1$.

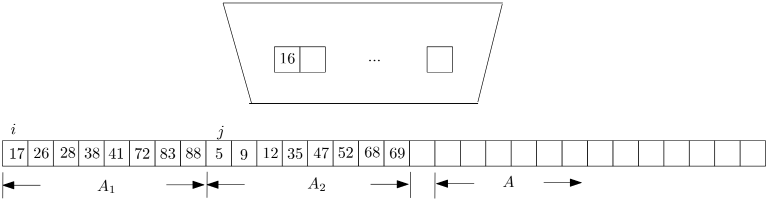
Repeat until $i > n/2$ or $j > n/2$:

1. If $A_1[i]$ (i.e., the i-th integer of A_1) is smaller than $A_2[j]$, append $A_1[i]$ to A, and increase i by 1.
2. Otherwise, append $A_2[j]$ to A, and increase j by 1.

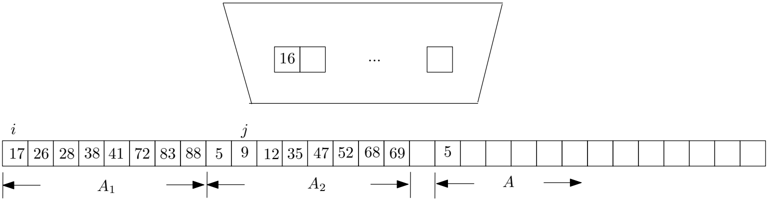
Think: What happens if $i > n/2$? What will you do to complete the merging?

Example

At the beginning of merging:

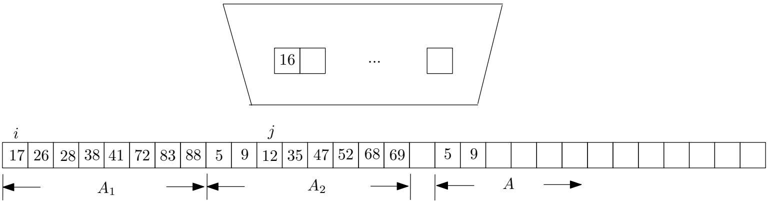


Appending 5 to A:

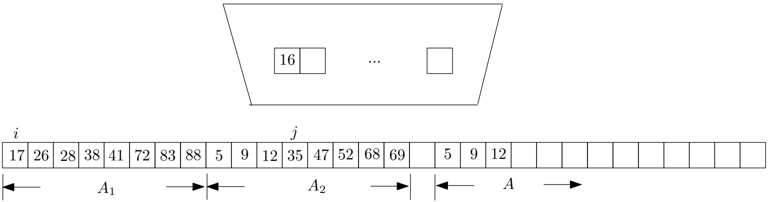


Example

Appending 9 to A:

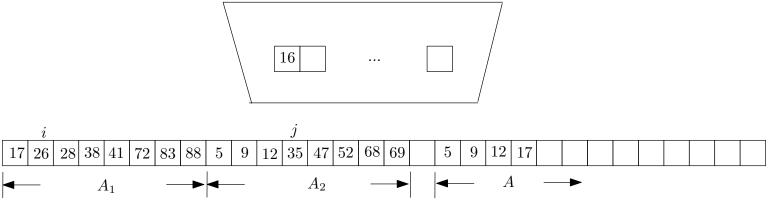


Appending 12 to A:



Example

Appending 17 to A:



And so on.

Running Time of Merge Sort

Let $f(n)$ denote the worst-case running time of merge sort when executed on an array of size n.

For $n = 1$, we have:

$$f(n) = O(1)$$

For $n \geq 1$:

$$f(n) \leq 2f(\lceil n/2 \rceil) + O(n)$$

where the term $2f(\lceil n/2 \rceil)$ is because the recursion sorts two arrays each of size at most $\lceil n/2 \rceil$, and the term $O(n)$ is the time of merging.

Running Time of Merge Sort

So it remains to solve the following recurrence:

$$\begin{aligned} f(1) &\leq c_1 \\ f(n) &\leq 2f(n/2) + c_2 n \end{aligned}$$

where c_1, c_2 are constants (whose values we do not care). If n is a power of 2, using the expansion method, we have:

$$\begin{aligned} f(n) &\leq 2f(n/2) + c_2 n \\ &\leq 2(2f(n/4) + c_2 n/2) + c_2 n = 4f(n/4) + 2c_2 n \\ &\leq 4(2f(n/8) + c_2 n/4) + 2c_2 n = 8f(n/8) + 3c_2 n \\ &\dots \\ &\leq 2^i f(n/2^i) + i \cdot c_2 n \\ &\dots \\ (h = \log_2 n) &\leq 2^h f(1) + h \cdot c_2 n \\ &\leq n \cdot c_1 + c_2 n \cdot \log_2 n = O(n \log n). \end{aligned}$$

Running Time of Merge Sort

How to remove the assumption that n is a power of 2? Hint: The rounding approach discussed in a previous lecture.