Lecture Notes: Matrix Similarity Transformation

Yufei Tao
Department of Computer Science and Engineering
Chinese University of Hong Kong
taoyf@cse.cuhk.edu.hk

In this lecture, we will introduce an important technique on matrices called *similarity transformation*. This technique is especially powerful in computing a high power of a matrix. Also, the technique will allow you to start appreciating the usefulness of eigenvalues and eigenvectors.

1 Matrix Similarity

Let us start by defining similar matrices:

Definition 1. *Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be $n \times n$ matrices. If we can find a non-singular $n \times n$ matrix $\boldsymbol{P}$ such that*

$$\boldsymbol{A} = \boldsymbol{P}^{-1}\boldsymbol{B}\boldsymbol{P} \tag{1}$$

then we say that $\boldsymbol{A}$ and $\boldsymbol{B}$ are **similar** *to each other.*

Note that (1) implies

$$\begin{aligned} \boldsymbol{P}\boldsymbol{A}\boldsymbol{P}^{-1} &= \boldsymbol{P}\boldsymbol{P}^{-1}\boldsymbol{B}\boldsymbol{P}\boldsymbol{P}^{-1} \Rightarrow \\ \boldsymbol{P}\boldsymbol{A}\boldsymbol{P}^{-1} &= \boldsymbol{I}\boldsymbol{B}\boldsymbol{I} \Rightarrow (\boldsymbol{I} \text{ is the } n \times n \text{ identity matrix}) \\ \boldsymbol{B} &= \boldsymbol{P}\boldsymbol{A}\boldsymbol{P}^{-1} \end{aligned}$$

In other words, in declaring matrix similarity, it does not matter which matrix ($\boldsymbol{A}$ or $\boldsymbol{B}$) is on the left hand side, and which gets multiplied with two other matrices.

There are numerous reasons why $\boldsymbol{A}$ and $\boldsymbol{B}$ are called similar. The following are two of them:

Lemma 1. *If $\boldsymbol{A}$ and $\boldsymbol{B}$ are similar, then they have the same rank.*

Proof. In general, let $\boldsymbol{C}$ be an $n \times n$ matrix with rank n. Then, both $\boldsymbol{C}\boldsymbol{A}$ and $\boldsymbol{A}\boldsymbol{C}$ have the same rank as $\boldsymbol{A}$ (the proof of this statement is left to you; hint: linear transformation and $\boldsymbol{C}$ has an inverse). Then, the lemma follows from the fact that both $\boldsymbol{P}$ and $\boldsymbol{P}^{-1}$ have rank n. □

Lemma 2. *If $\boldsymbol{A}$ and $\boldsymbol{B}$ are similar, then their characteristic equations imply each other; and hence, $\boldsymbol{A}$ and $\boldsymbol{B}$ have exactly the same eigenvalues.*

Proof. By symmetry, we will only show that the characteristic equation of $\boldsymbol{A}$ implies that of $\boldsymbol{B}$, namely, $det(\boldsymbol{A} - \lambda\boldsymbol{I}) = 0$ implies $det(\boldsymbol{B} - \lambda\boldsymbol{I}) = 0$. In fact:

$$\begin{aligned} det(\boldsymbol{A} - \lambda\boldsymbol{I}) &= 0 \Rightarrow \\ det(\boldsymbol{P}^{-1}\boldsymbol{B}\boldsymbol{P} - \lambda\boldsymbol{P}^{-1}\boldsymbol{I}\boldsymbol{P}) &= 0 \Rightarrow \\ det(\boldsymbol{P}^{-1}\boldsymbol{B}\boldsymbol{P} - \boldsymbol{P}^{-1}(\lambda\boldsymbol{I})\boldsymbol{P}) &= 0 \Rightarrow \\ det(\boldsymbol{P}^{-1}(\boldsymbol{B} - \lambda\boldsymbol{I})\boldsymbol{P}) &= 0 \Rightarrow \\ det(\boldsymbol{P}^{-1}) \cdot det(\boldsymbol{B} - \lambda\boldsymbol{I}) \cdot det(\boldsymbol{P}) &= 0 \Rightarrow \end{aligned}$$

Since $det(\boldsymbol{P}^{-1}) \neq 0$ and $det(\boldsymbol{P}) \neq 0$ (actually, we have $det(\boldsymbol{P}^{-1})det(\boldsymbol{P}) = 1$), the above leads to $det(\boldsymbol{B} - \lambda\boldsymbol{I}) = 0$. □

As mentioned earlier, matrix similarity is useful in computing a high power of a matrix. This is achieved by using the property below:

Lemma 3. *Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be similar matrices. For any integer $t \geq 1$, it holds that*

$$\boldsymbol{A}^t = \boldsymbol{P}^{-1}\boldsymbol{B}^t\boldsymbol{P}.$$

Proof.

$$\begin{aligned} \boldsymbol{A}^2 &= (\boldsymbol{P}^{-1}\boldsymbol{B}\boldsymbol{P})(\boldsymbol{P}^{-1}\boldsymbol{B}\boldsymbol{P}) \\ &= \boldsymbol{P}^{-1}\boldsymbol{B}\boldsymbol{I}\boldsymbol{B}\boldsymbol{P} \\ &= \boldsymbol{P}^{-1}\boldsymbol{B}^2\boldsymbol{P} \end{aligned}$$

$$\begin{aligned} \boldsymbol{A}^3 &= (\boldsymbol{P}^{-1}\boldsymbol{B}^2\boldsymbol{P})(\boldsymbol{P}^{-1}\boldsymbol{B}\boldsymbol{P}) \\ &= \boldsymbol{P}^{-1}\boldsymbol{B}^2\boldsymbol{I}\boldsymbol{B}\boldsymbol{P} \\ &= \boldsymbol{P}^{-1}\boldsymbol{B}^3\boldsymbol{P} \end{aligned}$$

Extending the argument to general t proves the lemma. □

Therefore, instead of computing $\boldsymbol{A}^t$, we could instead compute $\boldsymbol{B}^t$, provided that the latter is easier to work with. What kind of $\boldsymbol{B}$ would allow us to compute $\boldsymbol{B}^t$ quickly? An answer is: diagonal matrices, as shown in the next section.

2 Diagonal Matrices

Let $\boldsymbol{D}$ be an $n \times n$ diagonal matrix, namely, any entry of D not on the main diagonal of D is 0. Sometimes, we may use $diag[d_1, d_2, ..., d_n]$ as a short form for a diagonal matrix, where d_i ($1 \leq i \leq n$) is the element at the i-th row of the diagonal of D, namely:

$$diag[d_1, d_2, ..., d_n] = \begin{bmatrix} d_1 & 0 & ... & 0 \\ 0 & d_2 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & d_n \end{bmatrix}$$

Computation on diagonal matrices is often fairly easy. Let $\boldsymbol{A} = [a_{ij}]$ be an $n \times n$ matrix. Then:

$$\begin{bmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ ... & ... & ... & ... \\ a_{n1} & a_{n2} & ... & a_{nn} \end{bmatrix} \begin{bmatrix} d_1 & 0 & ... & 0 \\ 0 & d_2 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & d_n \end{bmatrix} = \begin{bmatrix} d_1 a_{11} & d_2 a_{12} & ... & d_n a_{1n} \\ d_1 a_{21} & d_2 a_{22} & ... & d_n a_{2n} \\ ... & ... & ... & ... \\ d_1 a_{n1} & d_2 a_{n2} & ... & d_n a_{nn} \end{bmatrix}$$

The effect of the multiplication is essentially to multiple the i-th $(1 \le i \le n)$ column of $\boldsymbol{A}$ by d_i. Likewise:

$$\begin{bmatrix} d_1 & 0 & ... & 0 \\ 0 & d_2 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & d_n \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ ... & ... & ... & ... \\ a_{n1} & a_{n2} & ... & a_{nn} \end{bmatrix} = \begin{bmatrix} d_1 a_{11} & d_1 a_{12} & ... & d_1 a_{1n} \\ d_2 a_{21} & d_2 a_{22} & ... & d_2 a_{2n} \\ ... & ... & ... & ... \\ d_n a_{n1} & d_n a_{n2} & ... & d_n a_{nn} \end{bmatrix}$$

The effect is essentially to multiple the i-th $(1 \le i \le n)$ row of $\boldsymbol{A}$ by d_i.

Further, powers of $\boldsymbol{D}$ are very easy to obtain. Specifically, if $\boldsymbol{D} = diag[d_1, d_2, ..., d_n]$, then for any integer $t \ge 1$, it holds that

$$\boldsymbol{D}^t = diag[d_1^t, d_2^t, ..., d_n^t].$$

Another wonderful property of a diagonal matrix is that its eigenvalues are trivial to acquire:

Lemma 4. *The eigenvalues of $\boldsymbol{D} = diag[d_1, d_2, ..., d_n]$ are precisely $d_1, d_2, ..., d_n$.*

Proof. The characteristic equation of $\boldsymbol{D}$ is

$$\begin{aligned} det(\boldsymbol{D} - \lambda \boldsymbol{I}) &= 0 \Rightarrow \\ (\lambda - d_1)(\lambda - d_2)...(\lambda - d_n) &= 0. \end{aligned}$$

The lemma thus follows. □

3 Similarity Transformation to a Diagonal Matrix

Henceforth, we will focus on only a special type of similarity transformation. Look at Definition 1 again. Given a matrix $\boldsymbol{A}$, we will strive to find a diagonal matrix to serve as the matrix $\boldsymbol{B}$. An important reason why we want to do so is that, as mentioned earlier, it allows us to compute $\boldsymbol{A}^t$ easily.

Example 1. Consider

$$\boldsymbol{A} = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 0 & 1 \\ 0 & 4 & 0 \end{bmatrix}$$

Later, we will show:

$$\boldsymbol{A} = \begin{bmatrix} 1 & 5 & 3 \\ 0 & 3 & 1 \\ 0 & -6 & 2 \end{bmatrix} diag[1, -2, 2] \begin{bmatrix} 1 & -7/3 & -1/3 \\ 0 & 1/6 & -1/12 \\ 0 & 1/2 & 1/4 \end{bmatrix}$$

Therefore,

$$A^t = \begin{bmatrix} 1 & 5 & 3 \\ 0 & 3 & 1 \\ 0 & -6 & 2 \end{bmatrix} diag[1, (-2)^t, 2^t] \begin{bmatrix} 1 & -7/3 & -1/3 \\ 0 & 1/6 & -1/12 \\ 0 & 1/2 & 1/4 \end{bmatrix}$$

□

Given $\boldsymbol{A}$, we refer to the process of finding a diagonal matrix $\boldsymbol{B}$ as a *diagonalization* of $\boldsymbol{A}$ (in Example 1, $\boldsymbol{B} = diag[1, 2, -2]$). If such $\boldsymbol{B}$ exists, we say that $\boldsymbol{A}$ is *diagonalizable*. Unfortunately, not all the $n \times n$ matrices are diagonalizable. The next lemma gives an if-and-only-if condition for a matrix to be diagonalizable.

Lemma 5. *An $n \times n$ matrix $\boldsymbol{A}$ is diagonalizable if and only if $\boldsymbol{A}$ has n linearly independent eigenvectors $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, ..., $\boldsymbol{v}_n$.*

Although we will not present a formal proof of the lemma, we give a precise procedure to diagonalize $\boldsymbol{A}$ when it is possible to do so (this procedure essentially proves the if-direction; the only-if direction follows similar ideas, and is left to you). As stated in the lemma, $\boldsymbol{A}$ needs to have n linearly independent eigenvectors $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, ..., $\boldsymbol{v}_n$. Let $\lambda_1, \lambda_2, ..., \lambda_n$ be the eigenvalues that $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, ..., $\boldsymbol{v}_n$ correspond to, respectively. Then, we construct:

- an $n \times n$ matrix $\boldsymbol{Q}$ by placing $\boldsymbol{v_i}$ as the i-th column of $\boldsymbol{Q}$ ($1 \le i \le n$).
- an $n \times n$ diagonal matrix $\boldsymbol{B} = diag[\lambda_1, \lambda_2, ..., \lambda_n]$.

The above construction definitely ensures that $\boldsymbol{A} = \boldsymbol{QBQ}^{-1}$ (if you insist on the form in Definition 1, set $\boldsymbol{P} = \boldsymbol{Q}^{-1}$), as illustrated in the following example.

Example 2. Consider again the matrix $\boldsymbol{A}$ in Example 1. Its characteristic equation is $(\lambda-1)(\lambda+2)(\lambda-2) = 0$. Hence, $\boldsymbol{A}$ has eigenvalues $\lambda_1 = 1$, $\lambda_2 = -2$, and $\lambda_3 = 2$.

For eigenvalue $\lambda_1 = 1$, an eigenvector is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. For eigenvalue $\lambda_2 = -2$, an eigenvector is $\begin{bmatrix} 5 \\ 3 \\ -6 \end{bmatrix}$. For eigenvalue $\lambda_3 = 2$, an eigenvector is $\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$. These 3 eigenvectors are linearly independent.

Therefore, by the diagonalization method described earlier, we have:

$$\boldsymbol{A} = \boldsymbol{Q}\, diag[1, -2, 2]\, \boldsymbol{Q}^{-1}$$

where

$$\boldsymbol{Q} = \begin{bmatrix} 1 & 5 & 3 \\ 0 & 3 & 1 \\ 0 & -6 & 2 \end{bmatrix}$$

□

A remark is in order at this point. So we have seen that if $\boldsymbol{A}$ is diagonalizable, $\boldsymbol{A}^t$ can be computed easily. But then how to compute $\boldsymbol{A}^t$ when $\boldsymbol{A}$ is not diagonalizable? Unfortunately, this is not always easy; as far as this course is concerned, there are no clever tricks to do so.

4 A Sufficient Condition for Diagonalization

Although the diagonalization approach of the previous section is fairly effective, its application requires us to check whether the eigenvectors we have obtained are linearly independent. Next, we give a lemma that allows us to skip the check-up in some situations.

Lemma 6. *Suppose that $\boldsymbol{A}$ has n distinct eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$. Let $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, ..., $\boldsymbol{v}_n$ be eigenvectors of $\boldsymbol{A}$ corresponding to those eigenvalues, respectively. Then, $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, ..., $\boldsymbol{v}_n$ must be linearly independent.*

Proof. Suppose on the contrary that there are at most only $r < n$ linearly independent vectors among $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, ..., $\boldsymbol{v}_n$. Without loss of generality, assume the r vectors to be $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, ..., $\boldsymbol{v}_r$. Hence, $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, ..., $\boldsymbol{v}_{r+1}$ are linearly dependent, such that there exist real values $c_1, c_2, ..., c_{r+1}$ that are not all 0, and make the following equation hold

$$\begin{aligned} c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + ... + c_{r+1}\boldsymbol{v}_{r+1} &= \boldsymbol{0} \Rightarrow && (2) \\ c_1\boldsymbol{A}\boldsymbol{v}_1 + c_2\boldsymbol{A}\boldsymbol{v}_2 + ... + c_{r+1}\boldsymbol{A}\boldsymbol{v}_{r+1} &= \boldsymbol{0} \Rightarrow \\ c_1\lambda_1\boldsymbol{v}_1 + c_2\lambda_2\boldsymbol{v}_2 + ... + c_{r+1}\lambda_{r+1}\boldsymbol{v}_{r+1} &= \boldsymbol{0} && (3) \end{aligned}$$

From (2) and (3), we obtain:

$$c_1(\lambda_{r+1} - \lambda_1)\boldsymbol{v}_1 + c_2(\lambda_{r+1} - \lambda_2)\boldsymbol{v}_2 + ... + c_r(\lambda_{r+1} - \lambda_r)\boldsymbol{v_r} = \boldsymbol{0}.$$

The linear independence of $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, ..., $\boldsymbol{v}_r$ asserts that $c_1(\lambda_{r+1} - \lambda_1) = c_2(\lambda_{r+1} - \lambda_2) = ... = c_1(\lambda_{r+1} - \lambda_r) = 0$. As the eigenvalues are mutually different, we have $c_1 = c_2 = ... = c_r = 0$. However, looking at (2) now and using the fact $\boldsymbol{v}_{r+1} \neq \boldsymbol{0}$, we know that $c_{r+1} = 0$. This is a contradiction to $c_1, ..., c_{r+1}$ not all being 0. □

Example 3. In Example 2, the matrix $\boldsymbol{A}$ has distinct eigenvalues: 1, −2, 2. Let $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ be *any* eigenvectors corresponding to 1, −2, 2, respectively. By the above lemma, $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ *must* be linearly independent. Hence, we can definitely use $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ to perform diagonalization as described in Section 3. □

We should note, once again, that Lemma 6 is a sufficient condition. In other words, even if $\boldsymbol{A}$ does not have n distinct eigenvalues, it may still be possible that $\boldsymbol{A}$ can be diagonalized. We will see an example in the next section.

5 Symmetric Matrices are Always Diagonalizable

Recall that an $n \times n$ matrix $\boldsymbol{A}$ is symmetric if $\boldsymbol{A} = \boldsymbol{A}^T$. In this lecture, we will be satisfied by the following result, while later in the course we will see a stronger version of it:

Lemma 7. *A symmetric $n \times n$ matrix $\boldsymbol{A}$ definitely has n linearly independent eigenvectors.*

The lemma, combined with Lemma 5, indicates that a symmetric matrix can always be diagonalized, regardless of how many distinct eigenvalues $\boldsymbol{A}$ has.

Example 4. Consider

$$\boldsymbol{A} = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Its characteristic equation is $(\lambda-1)^2(\lambda+2) = 0$. Hence, $\boldsymbol{A}$ has two eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -2$.

For eigenvalue $\lambda_1 = 1$, all the eigenvectors can be represented as $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ satisfying:

$$x_1 = v - u, x_2 = u, x_3 = v$$

with $u, v \in \mathbb{R}$. Setting (u, v) to $(1, 0)$ and $(0, 1)$ respectively gives us two linearly independent eigenvectors:

$$\boldsymbol{x}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \boldsymbol{x}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

For eigenvalue $\lambda_2 = -2$, all the eigenvectors can be represented as $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ satisfying:

$$x_1 = -t, x_2 = -t, x_3 = t$$

with $t \in \mathbb{R}$. Setting $t = 1$ gives us another eigenvector:

$$\boldsymbol{x}_3 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

Vectors $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, and $\boldsymbol{x}_3$ are linearly independent. Therefore, by the diagonalization method in Section 3, we have:

$$\boldsymbol{A} = \boldsymbol{Q}\, diag[1, 1, -2]\, \boldsymbol{Q}^{-1}$$

where

$$\boldsymbol{Q} = \begin{bmatrix} -1 & 1 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

$\square$

Appendix: Example of a Non-Diagonalizable Matrix

Consider

$$\boldsymbol{A} \;=\; \begin{bmatrix} -1 & 1 & 0 \\ -4 & 3 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

Its characteristic equation is

$$(\lambda - 1)^2(\lambda - 2) \;=\; 0.$$

Hence, $\boldsymbol{A}$ has only 2 eigenvalues: $\lambda_1 = 1$ and $\lambda_2 = 2$.

Focus now on the eigenvalue $\lambda_1 = 1$. To find its corresponding eigenvectors $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, we solve the equation:

$$\begin{aligned}
(\boldsymbol{A} - \lambda_1 \boldsymbol{I})\boldsymbol{x} \;&=\; \mathbf{0} \;\Rightarrow \\
\begin{bmatrix} -2 & 1 & 0 \\ -4 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \;&=\; \mathbf{0} \;\Rightarrow \\
\begin{bmatrix} -2 & 1 & 0 \\ 0 & 1/2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \;&=\; \mathbf{0} \;\Rightarrow
\end{aligned}$$

Hence, we know that

$$\begin{aligned}
-2x_1 + x_2 \;&=\; 0 \\
(1/2)x_2 + x_3 \;&=\; 0
\end{aligned}$$

Therefore, the set of solutions to the above problem is the set $\mathit{EigenSpace}(\lambda_1)$ of vectors $\begin{bmatrix} -t \\ -2t \\ t \end{bmatrix}$ for all $t \in \mathbb{R}$. This set has dimension 1. This means that it is hopeless to pick 2 eigenvectors of λ_1 that are linearly independent.

You can also verify that it is impossible to pick 2 eigenvectors of λ_2 that are linearly independent (actually, this is quite obvious if you still remember that the geometric multiplicity of λ_2 cannot exceed its algebraic multiplicity, which is 1).

It follows that $\boldsymbol{A}$ has only 2 eigenvectors that are linearly independent. By Lemma 5, we know that $\boldsymbol{A}$ is not diagonalizable. $\square$