Lecture Notes: Line Integrals by Coordinate and by Dot Product

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Line integrals by arc length can be regarded as performing integration using a scalar function along a curve. Today we will discuss a different form of line integrals, which perform integration using a *vector* function along a curve. Next, we will take several steps — in Sections 1, 2, and 3, respectively — to define this form of integrals.

1 Line Integrals by One Coordinate

Let us first introduce a convention. Suppose that $f(x_1, x_2, ..., x_d)$ is a scalar function with real-valued parameters. Given a point $p = (x_1, x_2, ..., x_d)$ in $\mathbb{R}^d$, we use $f(p)$ as a short form for $f(x_1, x_2, ..., x_d)$.

Definition 1. *Let C be a smooth curve in $\mathbb{R}^d$ with a starting point and an ending point. Break C into a sequence of n curves $C_1, C_2, ..., C_n$ such that (i) C_1 has the same starting point as C, (ii) for $j \in [1, n-1]$, the ending point of C_j is the starting point of C_{j+1}, and (iii) C_n has the same ending point as C. Define ℓ to be the maximum length of $C_1, C_2, ..., C_n$. For each $j \in [1, n]$:*

- *choose an arbitrary point p_j on C_j*
- *denote by $\Delta_1[j] = x'_1[j] - x_1[j]$ where $x_1[j]$ and $x'_1[j]$ are the x_1-coordinates of the starting and ending points of C_j, respectively.*

For a scalar function $f(x_1, x_2, ..., x_d)$, if the following limit exists:

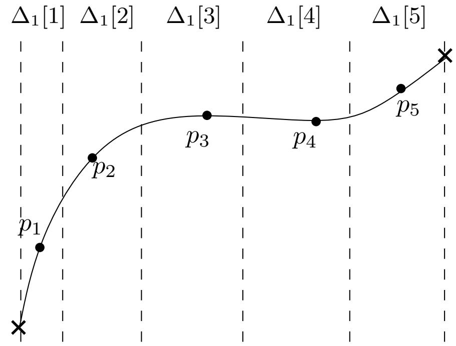
$$\lim_{\ell \to 0} \sum_{j=1}^{n} f(p_j) \cdot \Delta_1[j]$$

then we define

$$\int_C f(x_1, ..., x_d)\, dx_1$$

to be the above limit.

The figure below illustrates the curve partitioning in the above definition for $n = 5$ where x_1 refers to the horizontal dimension:



Note that as ℓ tends to 0, n tends to ∞. We state the next intuitive lemma without proof:

Lemma 1. *Suppose that the curve C in Definition 2 is defined by $\boldsymbol{r}(t) = [x_1(t), x_2(t), ..., x_d(t)]$ with $t \in [t_1, t_2]$. When $f(x_1(t), x_2(t), ..., x_d(t))$ is continuous in $[t_1, t_2]$, it holds that*

$$\int_C f(x_1, ..., x_d)\, dx_1 \quad = \quad \int_{t_1}^{t_2} f(x_1(t), ..., x_d(t)) \frac{dx_1}{dt} dt.$$

Example 1. Consider the circle $x^2 + y^2 = 1$. Let C be the arc from point $q_1 = (\sqrt{3}/2, 1/2)$ counterclockwise to point $q_2 = (1/2, \sqrt{3}/2)$. Calculate $\int_C \frac{1}{y}\, dx$.

Solution. The circle can be represented with $\boldsymbol{r}(t) = [x(t), y(t)]$ where $x(t) = \cos(t)$ and $y(t) = \sin(t)$. q_1 and q_2 correspond to $\boldsymbol{r}(\pi/6)$ and $\boldsymbol{r}(\pi/3)$, respectively. Hence, we have:

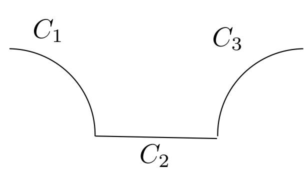
$$\begin{aligned}
\int_C \frac{1}{y}\, dx &= \int_{\pi/6}^{\pi/3} \frac{1}{y} \frac{dx}{dt} dt \\
&= \int_{\pi/6}^{\pi/3} \frac{1}{\sin(t)} \cdot (-\sin(t))\, dt \\
&= \int_{\pi/6}^{\pi/3} -1\, dt = -\pi/6.
\end{aligned}$$

□

Definition 2 requires that C should be smooth. Suppose that C is not a smooth curve, but can be broken into a *finite* number of smooth curves $C_1, C_2, ..., C_k$ (for some k). We say that C is *piecewise smooth.* For such a curve C, we define

$$\int_C f(x_1, ..., x_d)\, dx_1 \quad = \quad \sum_{i=1}^{k} \int_{C_i} f(x_1(t), ..., x_d(t))\, dx_1.$$

For example, in the figure below, let curve C be the concatenation of C_1, C_2 and C_3. C is not smooth, but is piecewise smooth.



Although the statement of Definition 2 is about coordinate x_1, we can define $\int_C f(x_1, ..., x_d)\, dx_i$ for any coordinate x_i with $i \in [1, d]$ in the same manner.

2 Line Integrals by All Coordinates

Suppose that we are given d scalar functions $f_1(x_1, ..., x_d), f_2(x_1, ..., x_d), ..., f_d(x_1, ..., x_d)$. Let C be a smooth curve in $\mathbb{R}^d$ from point p to point q. Also, let $\boldsymbol{r}(t) = [x_1(t), x_2(t), ..., x_d(t)]$ be a parametric form of C, such that p and q are given by $t = t_p$ and $t = t_q$, respectively.

From our earlier discussion, when all of $f_1(x_1(t), ..., x_d(t)), f_2(x_1(t), ..., x_d(t)), ..., f_d(x_1(t), ..., x_d(t))$ are continuous in $[t_p, t_q]$, it holds that

$$\begin{aligned}
& \int_C f_1(x_1, ..., x_d)\, dx_1 + \int_C f_2(x_2, , ..., x_d)\, dx_2 + ... + \int_C f_d(x_d, , ..., x_d)\, dx_d \\
= & \int_{t_p}^{t_q} \left(f_1(x_1(t), ..., x_d(t)) \frac{dx_1}{dt} + f_2(x_1(t), ..., x_d(t)) \frac{dx_2}{dt} + ... + f_d(x_1(t), ..., x_d(t)) \frac{dx_d}{dt} \right) dt.
\end{aligned} \tag{1}$$

Example 2. Consider the circle $x^2 + y^2 = 1$. Let C be the arc from $q_1 = (\sqrt{3}/2, 1/2)$ counter-clockwise to point $q_2 = (1/2, \sqrt{3}/2)$. Calculate

$$\int_C \frac{1}{y}\, dx + \int_C \frac{y}{x}\, dy.$$

Solution. The circle can be represented with $\boldsymbol{r}(t) = [x(t), y(t)]$ where $x(t) = \cos(t)$ and $y(t) = \sin(t)$. Points p and q correspond to $\boldsymbol{r}(\pi/6)$ and $\boldsymbol{r}(\pi/3)$, respectively. Hence, we have:

$$\begin{aligned}
\int_C \frac{1}{y}\, dx + \int_C \frac{y}{x}\, dy &= \int_{\pi/6}^{\pi/3} \frac{1}{y} \frac{dx}{dt} dt + \int_{\pi/6}^{\pi/3} \frac{y}{x} \frac{dy}{dt} dt \\
&= \int_{\pi/6}^{\pi/3} \frac{1}{\sin(t)} \cdot (-\sin(t))\, dt + \int_{\pi/6}^{\pi/3} \frac{\sin(t)}{\cos(t)} \cdot \cos(t)\, dt \\
&= \int_{\pi/6}^{\pi/3} -1\, dt + \int_{\pi/6}^{\pi/3} \sin(t) dt \\
&= -\frac{\pi}{6} + \frac{\sqrt{3} - 1}{2}.
\end{aligned}$$

□

3 Line Integrals by Dot Product

We are ready to define how to perform integration along a curve using a vector function. For this purpose, let us introduce another convention. Suppose that $\boldsymbol{f}(x_1, x_2, ..., x_d)$ is a scalar function with real-valued parameters. Given a point $p = (x_1, x_2, ..., x_d)$ in $\mathbb{R}^d$, we use $\boldsymbol{f}(p)$ as a short form for $\boldsymbol{f}(x_1, x_2, ..., x_d)$.

Definition 2. *Let:*

- $\boldsymbol{f}(x_1, ..., x_d)$ *be a vector function whose output is a d-dimensional vector*
- $\boldsymbol{r}(t)$ *be a smooth d-dimensional curve, and*
- C *be an arc on the curve with a starting point and an ending point.*

Break C into a sequence of n curves $C_1, C_2, ..., C_n$ such that (i) C_1 has the same starting point as C, (ii) for $j \in [1, n-1]$, the ending point of C_j is the starting point of C_{j+1}, and (iii) C_n has the same ending point as C. Define ℓ to be the maximum length of $C_1, C_2, ..., C_n$. For each $j \in [1, n]$:

- *choose an arbitrary point p_j on C_j*
- *denote by $\mathbf{\Delta}[j]$ be the vector defined by the directed segment pointing from the starting point of C_j to the ending point of C_j.*

If the following limit exists:

$$\lim_{\ell \to 0} \sum_{j=1}^{n} \boldsymbol{f}(p_j) \cdot \mathbf{\Delta}[j]$$

then we define

$$\int_C \boldsymbol{f}(\boldsymbol{r}) \cdot d\boldsymbol{r} \tag{2}$$

to be the above limit.

Although the above definition may look a bit complicated, it is essentially the same as line integrals by "all coordinates". To see this, write out the components of $\boldsymbol{f}(x_1, ..., x_d)$ and $\mathbf{\Delta}[j]$ as:

$$\begin{aligned}
\boldsymbol{f}(x_1, ..., x_d) &= [f_1(x_1, ..., x_d), ..., f_d(x_1, ..., x_d)] \\
\mathbf{\Delta}[j] &= [\Delta_1[j], ..., \Delta_d[j]].
\end{aligned}$$

Then:

$$\begin{aligned}
\int_C \boldsymbol{f}(\boldsymbol{r}) \cdot d\boldsymbol{r} &= \lim_{\ell \to 0} \sum_{j=1}^{n} \boldsymbol{f}(p_j) \cdot \mathbf{\Delta}[j] \\
&= \lim_{\ell \to 0} \sum_{j=1}^{n} \Big(f_1(p_j) \cdot \Delta_1[j] + ... + f_d(p_j) \cdot \Delta_d[j] \Big) \\
&= \Big(\lim_{\ell \to 0} \sum_{j=1}^{n} f_1(p_j) \cdot \Delta_1[j] \Big) + ... + \Big(\lim_{\ell \to 0} \sum_{j=1}^{n} f_d(p_j) \cdot \Delta_d[j] \Big) \\
&= \int_C f_1(x_1, ..., x_d)\, dx_1 + \int_C f_2(x_1, ..., x_d)\, dx_2 + ... + \int_C f_d(x_1, ..., x_d)\, dx_d
\end{aligned}$$

Example 3. Define:

$$\boldsymbol{f}(x, y) = \left[\frac{1}{y}, \frac{y}{x} \right].$$

Define a curve:

$$\boldsymbol{r}(t) = [\cos t, \sin t].$$

Let C be the arc on the above curve defined by increasing t from $\pi/6$ to $\pi/3$. Calculate $\int_C \boldsymbol{f}(\boldsymbol{r}) \cdot d\boldsymbol{r}$.

Solution. From the earlier discussion we know that

$$\begin{aligned}\int_C \boldsymbol{f}(\boldsymbol{r}) \cdot d\boldsymbol{r} &= \int_C \frac{1}{y}\,dx + \int_C \frac{y}{x}\,dy \\ &= \int_{\pi/6}^{\pi/3} \left(\frac{1}{y}\frac{dx}{dt} + \frac{y}{x}\frac{dy}{dt}\right) dt\end{aligned}$$

The rest of the derivation is the same as that in Example 2. □

The above example actually illustrates the following transformation:

$$\begin{aligned}\int_C \boldsymbol{f}(\boldsymbol{r}) \cdot d\boldsymbol{r} &= \int_C f_1(x_1, ..., x_d)\,dx_1 + ... + \int_C f_d(x_1, ..., x_d)\,dx_d \\ &= \int_C \left(f_1(x_1, ..., x_d)\,\frac{dx_1}{dt} + ... + f_d(x_1, ..., x_d)\,\frac{dx_d}{dt}\right) dt \\ &= \int_C [f_1(x_1, ..., x_d), ..., f_d(x_1, ..., x_d)] \cdot \left[\frac{dx_1}{dt}, ..., \frac{dx_d}{dt}\right] dt \\ &= \int_C \boldsymbol{f}(x_1(t), ..., x_d(t)) \cdot \boldsymbol{r}'(t)\,dt.\end{aligned}$$

The above equation provides a neater way to calculate $\int_C \boldsymbol{f}(\boldsymbol{r}) \cdot d\boldsymbol{r}$, as shown in the example below.

Example 4. Let us re-calculate the line integral in Example 3:

$$\begin{aligned}\int_C \boldsymbol{f}(\boldsymbol{r}) \cdot d\boldsymbol{r} &= \int_{\pi/6}^{\pi/3} \boldsymbol{f}(x(t), y(t)) \cdot \boldsymbol{r}'(t)\,dt \\ &= \int_{\pi/6}^{\pi/3} \left[\frac{1}{\sin(t)}, \frac{\sin(t)}{\cos(t)}\right] \cdot [-\sin(t), \cos(t)]\ dt \\ &= \int_{\pi/6}^{\pi/3} -1 + \sin(t)\,dt \\ &= -\frac{\pi}{6} + \frac{\sqrt{3}-1}{2}.\end{aligned}$$

□