Exercises: Surfaces

Problem 1. Consider the sphere $(x-1)^2 + (y-2)^2 + z^2 = 6$.

1. Give a normal vector of the sphere at point $(2, 2+\sqrt{2}, \sqrt{3})$.
2. Give the equation of the tangent plane at point $(2, 2+\sqrt{2}, \sqrt{3})$.

Solution:

1. Define $f(x,y,z) = (x-1)^2 + (y-2)^2 + z^2 - 6$. Its gradient is

$$\begin{aligned} \nabla f(x,y,z) &= \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right] \\ &= [2(x-1), 2(y-2), 2z]\,. \end{aligned}$$

   Hence, $\nabla f(2, 2+\sqrt{2}, \sqrt{3}) = [2, 2\sqrt{2}, 2\sqrt{3}]$ is a normal vector at point $(2, 2+\sqrt{2}, \sqrt{3})$.

2. At this stage, you should be able to write out the equation of the plane directly (by resorting to dot product):

$$2(x-2) + 2\sqrt{2}(y-2-\sqrt{2}) + 2\sqrt{3}(z-\sqrt{3}) = 0.$$

Problem 2. As before, consider the sphere $(x-1)^2 + (y-2)^2 + z^2 = 6$.

1. Let C_1 be the curve on the sphere satisfying $x = 2$. Give a tangent vector $\boldsymbol{v}_1$ of C_1 at point $(2, 2+\sqrt{2}, \sqrt{3})$.
2. Let C_2 be the curve on the sphere satisfying $y = 2+\sqrt{2}$. Give a tangent vector $\boldsymbol{v}_2$ of C_2 at point $(2, 2+\sqrt{2}, \sqrt{3})$.
3. Compute $\boldsymbol{v}_1 \times \boldsymbol{v}_2$.

Solution:

1. Let C_1' be the part of C_1 satisfying $z \geq 0$. Let us write C_1' into its parametric form $\boldsymbol{r}(t) = [x(t), y(t), z(t)]$.

$$\begin{aligned} x(t) &= 2 \\ y(t) &= t \\ z(t) &= \sqrt{5-(t-2)^2}. \end{aligned}$$

   Hence, $\boldsymbol{r}'(t) = [0, 1, \frac{2-t}{\sqrt{5-(t-2)^2}}]$. Point $(2, 2+\sqrt{2}, \sqrt{3})$ is given by $t = 2+\sqrt{2}$. Hence, a tangent vector is $\boldsymbol{r}'(2+\sqrt{2}) = [0, 1, -\sqrt{2/3}]$.

2. Let C_2' be the part of C_2 satisfying $z \geq 0$. Let us write C_2' into its parametric form $\boldsymbol{r}(t) = [x(t), y(t), z(t)]$.

$$\begin{aligned} x(t) &= t \\ y(t) &= 2+\sqrt{2} \\ z(t) &= \sqrt{4-(t-1)^2}. \end{aligned}$$

Hence, $\boldsymbol{r}'(t) = [1, 0, \frac{1-t}{\sqrt{4-(t-1)^2}}]$. Point $(2, 2+\sqrt{2}, \sqrt{3})$ is given by $t = 2$. Hence, a tangent vector is $\boldsymbol{r}'(2+\sqrt{2}) = [1, 0, -\sqrt{1/3}]$.

3.

$$[0, 1, -\sqrt{2/3}] \times [1, 0, -\sqrt{1/3}] \quad = \quad [-\sqrt{1/3}, -\sqrt{2/3}, -1].$$

By the geometric property of cross product, this is another normal vector to the sphere at $(2, 2+\sqrt{2}, \sqrt{3})$.

Problem 3. Sphere $(x-1)^2 + (y-2)^2 + z^2 = 6$ can also be represented in the parametric form:

$$\begin{aligned} x(u,v) &= 1 + \sqrt{6}\cos(u) \\ y(u,v) &= 2 + \sqrt{6}\sin(u)\cos(v) \\ z(u,v) &= \sqrt{6}\sin(u)\sin(v) \end{aligned}$$

By fixing v to the value satisfying $\cos(v) = \sqrt{2/5}$ and $\sin(v) = \sqrt{3/5}$, from the above we get a curve C on the sphere that passes point $p = (2, 2+\sqrt{2}, \sqrt{3})$. Give a tangent vector of C at the point.

Solution: C has the parametric form $\boldsymbol{r}(u) = [x(u), y(u), z(u)]$ where:

$$\begin{aligned} x(u) &= 1 + \sqrt{6}\cos(u) \\ y(u) &= 2 + \sqrt{6}\frac{\sqrt{2}}{\sqrt{5}}\sin(u) = 2 + \frac{\sqrt{12}}{\sqrt{5}}\sin(u) \\ z(u) &= \sqrt{6}\frac{\sqrt{3}}{\sqrt{5}}\sin(v) = \frac{\sqrt{18}}{\sqrt{5}}\sin(u) \end{aligned}$$

Hence, $\boldsymbol{r}'(u) = [-\sqrt{6}\sin(u), \frac{\sqrt{12}}{\sqrt{5}}\cos(u), \frac{\sqrt{18}}{\sqrt{5}}\cos(u)]$.

As C passes point p, we know

$$\begin{aligned} 1 + \sqrt{6}\cos(u) &= 2 \\ 2 + \frac{\sqrt{12}}{\sqrt{5}}\sin(u) &= 2 + \sqrt{2} \end{aligned}$$

giving $\cos(u) = \sqrt{1/6}$ and $\sin(u) = \sqrt{5/6}$. Hence, at p, a tangent vector is

$$\begin{aligned} \boldsymbol{r}'(u) &= [-\sqrt{6}\sin(u), \frac{\sqrt{12}}{\sqrt{5}}\cos(u), \frac{\sqrt{18}}{\sqrt{5}}\cos(u)] \\ &= [-\sqrt{6}\frac{\sqrt{5}}{\sqrt{6}}, \frac{\sqrt{12}}{\sqrt{5}}\frac{\sqrt{1}}{\sqrt{6}}, \frac{\sqrt{18}}{\sqrt{5}}\frac{\sqrt{1}}{\sqrt{6}}] \\ &= [-\sqrt{5}, \sqrt{2/5}, \sqrt{3/5}]. \end{aligned}$$

Problem 4. This problem is designed to show you how to use gradient to compute the normal vector of a tangle line in 2d space. Consider the circle $(x-1)^2 + (y-2)^2 = 5$. Give a vector whose direction is perpendicular to the tangent line of the circle at point $(2, 4)$.

Solution: Define $f(x,y) = (x-1)^2 + (y-2)^2 - 5$. The circle satisfies $f(x,y) = 0$.

Let us represent the circle in its parametric form $\boldsymbol{r}(t) = [x(t), y(t)]$. As we will see, we do need to worry about how to formulate $x(t)$ and $y(t)$ at all. It must hold that

$$f(x(t), y(t)) = 0$$

Taking the derivative of both sides with respect to t gives

$$\begin{aligned} \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} &= 0 \quad \Rightarrow \\ \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right] \cdot \left[\frac{dx}{dt}, \frac{dy}{dt}\right] &= 0 \Rightarrow \\ \nabla f(x,y) \cdot [x'(t), y'(t)] &= 0. \end{aligned}$$

Note that $[x'(t), y'(t)]$ is a tangent vector of the point $p(x,y)$ on the circle given by t. Hence, as long as $\nabla f(x,y)$ and $[x'(t), y'(t)]$ are not $\mathbf{0}$, $\nabla f(x,y)$ is a vector normal to the tangent vector.

In our problem, $\nabla f(x,y) = [2(x-1), 2(y-2)]$. Hence, $\nabla f(2,4) = [2,4]$ is a solution.