Dimensionality Reduction 1: Dominance Screening

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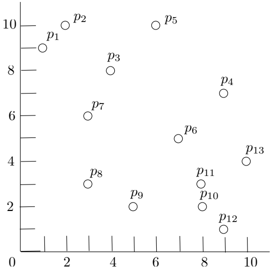
In this course, we focus on computational geometry problems in $\mathbb{R}^d$ where the dimensionality d is regarded as a constant. The **dimensionality reduction** technique works by reducing a problem of dimensionality d to one of dimensionality $d-1$. Today, we will apply the technique to tackle a problem called **dominance screening**.

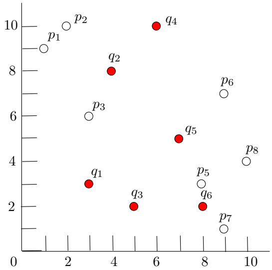
The Maxima Problem

A point $p_1 \in \mathbb{R}^d$ **dominates** $p_2 \in \mathbb{R}^d$ if the coordinate of p_1 is larger than or equal to that of p_2 in all dimensions, and strictly larger in at least one dimension.



Point p_5 dominates p_1, p_2, p_3, p_7, p_8, and p_9.

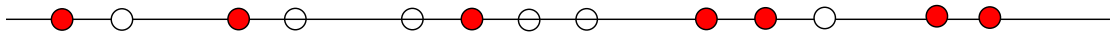
Let P and Q be sets of d-dimensional points in $\mathbb{R}^d$. In **dominance screening**, we want to report all the points in Q that are not dominated by any points in P. Set $n = |P| + |Q|$.



Suppose that P (or Q) is the set of white (or red, resp.) points. The result is $\{q_2, q_4\}$.

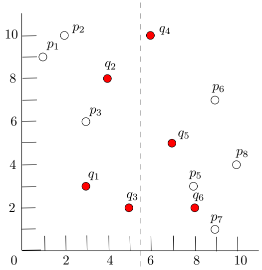
1D Dominance Screening

When $d = 1$, the problem can be easily solved in $O(n)$ time after sorting.



2D Dominance Screening

Divide the input into two halves by x-coordinate using a vertical line ℓ.

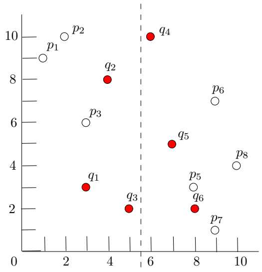


Let P_1 (resp., Q_1) be the set of white (resp., red) points on the left of ℓ. Define P_2 and Q_2 analogously with respect to the right of ℓ.

Remark: We will assume that such a line ℓ exists. Handling the opposite scenario is left to you.

2D Dominance Screening

We have two instances of dominance screening: the first on (P_1, Q_1), and the other on (P_2, Q_2).



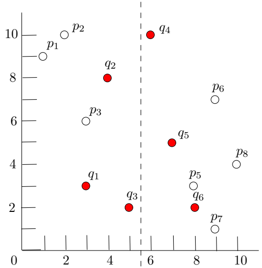
Solve each instance recursively.

The left instance reports $\{q_2, q_3\}$, and the right instance reports $\{q_4\}$.

Next, we will merge the two answers to obtain the final result.

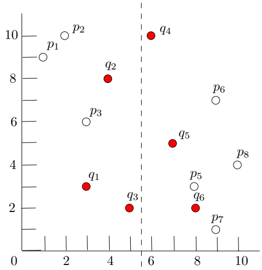
2D Dominance Screening

Observation 1: The right answer is definitely in the final result.
Observation 2: Let q be a point in the left answer. It is in the final result if and only if it is not dominated by any white point from the right instance.



2D Dominance Screening

We now resort to 1D dominance screening.



Let A_{left} be the left answer. Construct a 1D dominance screening problem with input sets P', Q' where

- P': obtained by projecting P_2 onto the y-axis
- Q': obtained by projecting A_{left} onto the y-axis.

2D Dominance Screening

Let us now analyze the running time. Let $f(n)$ be the time on $n = |P| + |Q|$ points. We have:

$$f(n) \quad \leq \quad 2 \cdot f(n/2) + O(n)$$

for $n \geq 2$.

> **Think:** Why is it "$+O(n)$", rather than "$+O(n \log n)$"?

Solving the recurrence gives: $f(n) = O(n \log n)$.

Dominance Screening in d-dimensional Space

1. Divide $P \cup Q$ into two equal halves by the first dimension. This yields two instances of d-dimensional dominance screening: (i) left instance (P_1, Q_1), and (ii) right instance (P_2, Q_2).
2. Solve the left and right instances, recursively. Let A_{left} and A_{right} be their answers, respectively.
3. Obtain a $(d-1)$-dimensional dominance screening problem (P', Q') where P' (resp., Q') is the projection of P_2 (resp., A_{left}) onto dimensions $2, 3, ..., d$. Solve this instance to obtain its answer A'.
4. Return $A_{right} \cup A'$.

Dominance Screening in d-dimensional Space

Let $f(n)$ be the time of our algorithm on n points. We have:

$$f(n) \leq 2 \cdot f(n/2) + g(n)$$

where $g(n)$ is the time of solving $(d-1)$-dimensional dominance screening. Solving the recurrence gives:

- when $d = 3$, $f(n) = O(n \log^2 n)$;
- when $d = 4$, $f(n) = O(n \log^3 n)$;
- ...
- in general, $f(n) = O(n \log^{d-1} n)$.

Remark: Using planesweep, we can solve the 3D problem in $O(n \log n)$ time (you will be guided to do so in an exercise). This immediately improves the time complexity to $O(n \log^{d-2} n)$ for $d \geq 3$.