CMSC5724: Quiz 2

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Problem 1 (30%). Given 2D points $p = (p[1], p[2])$ and $q = (q[1], q[2])$, define

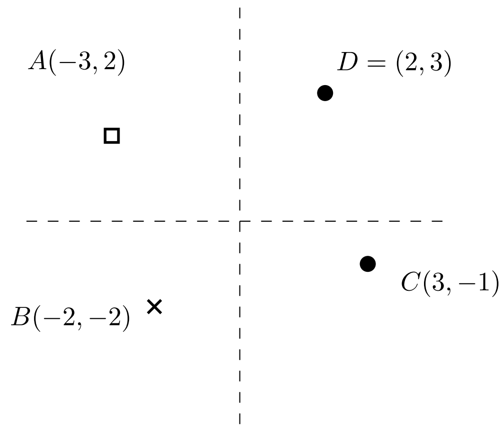
$$K(p, q) = 3 + 4p[2]q[2] + 2(p[1])^2(q[1])^2 + 16(p[1])^2(q[1])^2(p[2])^2(q[2])^2.$$

Prove: $K(p, q)$ is a kernel function. Specifically, you need to show a mapping function $\phi : \mathbb{R}^2 \to \mathbb{R}^d$ for some integer d such that $K(p, q)$ equals the dot product of $\phi(p)$ and $\phi(q)$.

Answer:

$$\phi(x) = (\sqrt{3}, 2x[2], \sqrt{2}(x[1])^2, 4(x[1]x[2])^2)$$

Problem 2 (30%). Consider a training set P including the points below



where the two dots have label 1, the box has label 2, and the cross has label 3. We have a 3-class linear classifier defined by vectors $\boldsymbol{w}_1 = (2, 0)$, $\boldsymbol{w}_2 = (-1, 1)$, and $\boldsymbol{w}_3 = (0, -1)$ (note that this classifier separates P). Calculate the margin of the classifier.

Answer: Let $W = \{w_1, w_2, w_3\}$.

$$\begin{aligned} margin(A \mid W) &= \min\{\frac{\boldsymbol{w_2} \cdot \boldsymbol{A} - \boldsymbol{w_1} \cdot \boldsymbol{A}}{\sqrt{2 \times \sum_{i=1}^{3} |\boldsymbol{w_i}|^2}}, \frac{\boldsymbol{w_2} \cdot \boldsymbol{A} - \boldsymbol{w_3} \cdot \boldsymbol{A}}{\sqrt{2 \times \sum_{i=1}^{3} |\boldsymbol{w_i}|^2}}\} = \min\{\frac{5-(-6)}{\sqrt{14}}, \frac{5-(-2)}{\sqrt{14}}\} \\ &= \frac{7}{\sqrt{14}} \end{aligned}$$

Similarly,

$$margin(B \mid W) = \min\{\frac{2-(-4)}{\sqrt{14}}, \frac{2-0}{\sqrt{14}}\} = \frac{2}{\sqrt{14}}$$

$$margin(C \mid W) = \min\{\frac{6-(-2)}{\sqrt{14}}, \frac{6-1}{\sqrt{14}}\} = \frac{5}{\sqrt{14}}$$

$$margin(D \mid W) = \min\{\frac{4-1}{\sqrt{14}}, \frac{4-(-3)}{\sqrt{14}}\} = \frac{3}{\sqrt{14}}$$

Therefore, the classifier's margin equals $\frac{2}{\sqrt{14}}$.

Problem 3 (40%). Suppose that $\mathcal{M}$ is a set of classifiers in $\mathbb{R}^2$. Each classifier $M \in \mathcal{M}$ is defined by $a, b, c \in \mathbb{R}$ such that for every $p \in \mathbb{R}^2$, $M(p)$ equals 1 if $a \cdot p[1] + b \cdot p[2] + c \geq 0$, or -1 otherwise.

1. Show the VC-dimension of $\mathcal{M}$ on $\mathbb{R}^2$. You do <u>not</u> need to provide a proof.
2. Consider $\mathcal{M}'$ as any subset of $\mathcal{M}$. Prove: the VC-dimension of $\mathcal{M}'$ on $\mathbb{R}^2$ is no more than the VC-dimension of $\mathcal{M}$ on $\mathbb{R}^2$.

Answer:

1. 3.
2. Proof. Since $\mathcal{M}' \subseteq \mathcal{M}$, if a set of points $P \subseteq \mathbb{R}^2$ is shattered by $\mathcal{M}'$, P is also shattered by $\mathcal{M}$, meaning that the VC-dimension of $\mathcal{M}$ on $\mathbb{R}^2$ is no less than the VC-dimension of $\mathcal{M}'$ on $\mathbb{R}^2$.