Association Rule Mining: Apriori

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Let U be a set of items, referred to as the **universe**.

An **itemset** I is a subset of U.
A k-**itemset** is an itemset of size k.

Let S be a collection of itemsets, each called a **transaction**.

The **support** of an itemset I is

$$support(I) = \left|\{T \in S \mid I \subseteq T\}\right|$$

namely, the number of transactions in S that contain I.

Example

- $U = \{$beer, bread, butter, milk, potato, onion$\}$
- The following table shows a collection of 5 transactions:

id	items
1	beer, bread
2	beer, butter
3	butter, milk, potato
4	beer, bread, butter, milk, onion
5	beer, bread, butter, milk
6	beer, bread, milk, onion

- If $I = \{beer, bread\}$, then $support(I) = 4$.

An **association rule** R has the form

$$I_1 \rightarrow I_2$$

where both I_1 and I_2 are non-empty itemsets satisfying $I_1 \cap I_2 = \emptyset$.

The **support** of R, denoted as $supp(R)$, equals the support of the itemset $I_1 \cup I_2$.

The **confidence** of R equals

$$conf(R) = \frac{support(I_1 \cup I_2)}{support(I_1)}.$$

Example

id	items
1	beer, bread
2	beer, butter
3	butter, milk, potato
4	beer, bread, butter, milk, onion
5	beer, bread, butter, milk
6	beer, bread, milk, onion

- The rule "$\{beer\} \rightarrow \{bread\}$" has support 4 and confidence 4/5.
- "$\{beer\} \rightarrow \{milk\}$" has support 3 and confidence 3/5.
- "$\{butter, potato\} \rightarrow \{milk\}$" has support 1 and confidence 1.

Problem (Association Rule Mining)

Given (i) a set S of transactions, and (ii) two values *minsup* and *minconf*, we want to find all the association rules R such that

$$\begin{aligned} supp(R) &\geq minsup \\ conf(R) &\geq minconf. \end{aligned}$$

Think:

- Why does it make sense to find such association rules?
- Why purposes do *minsup* and *minconf* serve?

Naively, we can solve the association rule problem by calculating the support and confidence values of all the possible association rules. But this method is prohibitively slow due to the large number of possible rules.

Next, we describe a better algorithm called **Apriori**.

Let I be an itemset. We say that I is **frequent** if $support(I) \geq minsup$.

Example

id	items
1	beer, bread
2	beer, butter
3	butter, milk, potato
4	beer, bread, butter, milk, onion
5	beer, bread, butter, milk
6	beer, bread, milk, onion

Assume that $minsup = 3$. Then:

- $\{beer\}$, $\{beer, bread\}$, and $\{beer, bread, milk\}$ are all frequent itemsets.
- $\{potato\}, \{potato, onion\}$, and $\{beer, milk, onion\}$ are not frequent itemsets.

If $I_1 \rightarrow I_2$ is an association rule that should be reported, by definition, it must hold that the itemset $I_1 \cup I_2$ is frequent.

Motivated by this observation, Apriori runs in two steps:

1. (Frequent itemsets computation): Report all the frequent itemsets of U.
2. (Rule generation): Generate association rules from the above frequent itemsets.

Next, we will explain each step in turn.

The next lemma is straightforward:

Lemma

$support(I_1 \cup I_2) \leq support(I_1)$.

The above is known as the **anti-monotone property**.

Corollary

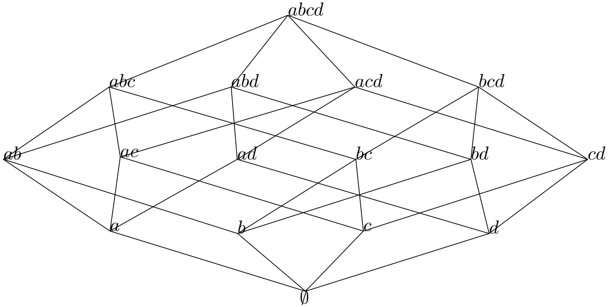
Suppose that $I_1 \subseteq I_2$.

- If I_2 is frequent, then I_1 must be frequent.
- If I_1 is not frequent, then I_2 cannot be frequent.

For example, if $\{beer, bread\}$ is frequent, then so must be $\{beer\}$ and $\{bread\}$. Conversely, if $\{beer\}$ is not frequent, then neither is $\{beer, bread\}$.

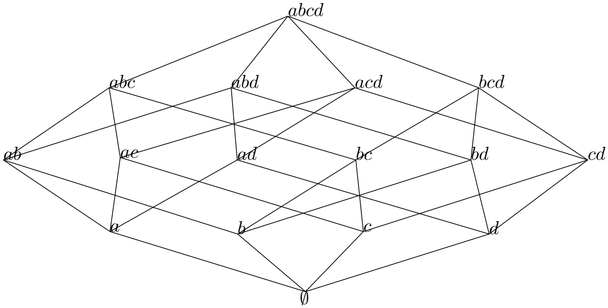
If the universal set U has n items, then there are $2^n - 1$ non-empty itemsets. It is helpful to think of these itemsets in the form of a lattice that captures the containment relation among these itemsets.

The figure below shows a lattice for $n = 4$ (assuming $U = \{a, b, c, d\}$). Note that an itemset I_1 is connected to an itemset I_2 of the upper level if and only if $I_1 \subset I_2$.



If we are unlucky, we may have to examine all the itemsets in the lattice. Fortunately, in reality, Corollary 6 implies a pruning rule for us to eliminate itemsets.

For example, if we already know that $\{a\}$ is infrequent, then we can immediately declare that all of $\{ab\}$, $\{ac\}$, $\{ad\}$, $\{abc\}$, $\{abd\}$, $\{acd\}$, and $\{abcd\}$ are infrequent.



Given an integer $k \in [1, n]$, let F_k denote the set of all frequent k-itemsets (i.e., itemsets of size k). Then, the entire set of frequent itemsets equals

$$F_1 \cup F_2 \cup ... \cup F_n.$$

Our earlier discussion indicates that, if $F_i = \emptyset$, then F_k is also empty for any $k > i$.

Therefore, Apriori adopts the following approach to find all the frequent itemsets:

1. $k = 1$
2. Find F_k. If $F_k = \emptyset$, terminate.
3. $k \leftarrow k + 1$; go to Line 2.

Next, we will clarify the details of Line 2.

Finding F_1.

Suppose that U has n items. Then, there are only n **candidate 1-itemsets**; let C_1 be the set of all these candidate itemsets. For each of them, calculate its support, and report the frequent ones.

Example

$$U = \{\text{beer, bread, butter, milk, potato, onion}\}$$

$$minsup = 3$$

id	items
1	beer, bread
2	beer, butter
3	butter, milk, potato
4	beer, bread, butter, milk, onion
5	beer, bread, butter, milk
6	beer, bread, milk, onion

- $C_1 = \{\{beer\}, \{bread\}, \{butter\}, \{milk\}, \{potato\}, \{onion\}\}$.
- $F_1 = \{\{beer\}, \{bread\}, \{butter\}, \{milk\}\}$.

Finding F_k ($k > 1$).

The main strategy is to identify a **candidate set** C_k of k-itemsets. Then, we can calculate the support of each such k-itemset, and report the frequent ones.

The key is to limit the size of C_k. Naively, we may set C_k to include all the k-itemsets, the number of which is $\binom{n}{k}$.

Next, we will discuss another method that generates a C_k whose size is usually much smaller.

First, impose an arbitrary total order on the items of U (e.g., the alphabetic order). Let $I = \{a_1, a_2, ..., a_k\}$ be a frequent k-itemset (i.e., an itemset in F_k). The lemma below is a straightforward corollary of Corollary 6:

Lemma

$\{a_1, a_2, ..., a_{k-2}, a_{k-1}\}$ and $\{a_1, a_2, ..., a_{k-2}, a_k\}$ are both frequent $(k-1)$-itemsets, namely, both of them need to be in F_{k-1}.

Next, given a $(k-1)$-itemset $I = \{b_1, b_2, ..., b_{k-2}, b_{k-1}\}$, we refer to the sequence $(b_1, b_2, ..., b_{k-2})$ as the prefix of I. Note that the prefix includes only the first $k-2$ items.

Motivated by this, Apriori generates C_k from F_{k-1} as follows.

1. Sort the itemsets in F_{k-1} by prefix. We will refer to the set of itemsets with the same prefix as a group.
2. Process each group as follows. For each pair of different itemsets $\{a_1, a_2, ..., a_{k-2}, a_{k-1}\}$ and $\{a_1, a_2, ..., a_{k-2}, a_k\}$ in the group, add to C_k the itemset $\{a_1, a_2, ..., a_k\}$.

Example

$$U = \{\text{beer, bread, butter, milk, potato, onion}\}$$
$$minsup = 3$$

id	items
1	beer, bread
2	beer, butter
3	butter, milk, potato
4	beer, bread, butter, milk, onion
5	beer, bread, butter, milk
6	beer, bread, milk, onion

- We know earlier $F_1 = \{\{beer\}, \{bread\}, \{butter\}, \{milk\}\}$.
- Hence, $C_2 = \{\{beer, bread\}, \{beer, butter\}, \{beer, milk\}, \{bread, butter\}, \{bread, milk\}, \{butter, milk\}\}$.
- Hence, $F_2 = \{\{beer, bread\}, \{beer, butter\}, \{beer, milk\}, \{bread, milk\}, \{butter, milk\}\}$.

Example

$$U = \{\text{beer, bread, butter, milk, potato, onion}\}$$
$$minsup = 3$$

id	items
1	beer, bread
2	beer, butter
3	butter, milk, potato
4	beer, bread, butter, milk, onion
5	beer, bread, butter, milk
6	beer, bread, milk, onion

- We know earlier $F_2 = \{\{beer, bread\}, \{beer, butter\}, \{beer, milk\}, \{bread, milk\}, \{butter, milk\}\}$.
- Hence, $C_3 = \{\{beer, bread, butter\}, \{beer, bread, milk\}, \{beer, butter, milk\}\}$.
- Hence, $F_3 = \{\{beer, bread, milk\}\}$.
- $C_4 = \emptyset$. Therefore, $F_4 = \emptyset$.

Recall that Apriori runs in two steps:

1. (Frequent itemsets computation): Report all the frequent itemsets of U.
2. (Rule generation): Generate association rules from the above frequent itemsets.

Next, we will explain the second step.

Let I be a frequent itemset with size $k \geq 2$. We first generate **candidate association rules** from I as follows. Divide I into disjoint non-empty itemsets I_1, I_2, namely, $I_1 \cup I_2 = I$ while $I_1 \cap I_2 = \emptyset$. Then, $I_1 \rightarrow I_2$ is taken as a candidate association rule.

As a second step, we compute the confidence values of all such candidate rules, and report those whose confidence values exceed *minconf*.

Note:

- *support*$(I_1 \rightarrow I_2)$ must be at least *minsup* (why?).
- To calculate the confidence of $I_1 \rightarrow I_2$, we need *support*(I) and *support*(I_1). Both values are directly available from the first step of Apriori (finding frequent itemsets), noticing that I_1 must be a frequent itemset.
- If I and I' are two frequent itemsets, no candidate rule generated from I can be identical to any candidate rule generated from I' (why?).

A drawback of the above method is that when k is large, it is quite expensive to compute the confidence values of $2^k - 2$ association rules. Next, we present a heuristic that can often reduce the number in practice.

As before, fix a frequent k-itemset I. Let I_1, I_2 be disjoint non-empty subsets of I with $I_1 \cup I_2 = I$. Similarly, let I_1', I_2' also be disjoint non-empty subsets of I with $I_1' \cup I_2' = I$. We have:

Lemma

If $I_1 \subset I_1'$, then $conf(I_1 \to I_2) \le conf(I_1' \to I_2')$.

We say that $I_1' \to I_2'$ **left-contains** $I_1 \to I_2$.

Proof.

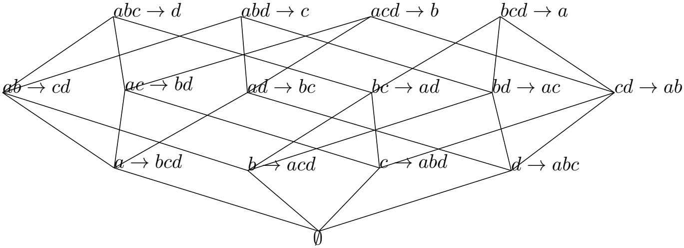
$$conf(I_1 \to I_2) = \frac{support(I)}{support(I_1)} \le \frac{support(I)}{support(I_1')} = conf(I_1' \to I_2').$$

□

Example

Suppose that $I = \{\{beer, bread, milk\}\}$. It must hold that $conf(\{beer, bread\} \to \{milk\}) \ge conf(\{beer\} \to \{milk, bread\})$.

We can organize all the candidate association rules generated from I in a lattice. The following figure illustrates the lattice for $I = \{abcd\}$. Note that a rule R_1 is connected to another rule R_2 of the upper level if and only if R_2 left-contains R_1.



Apriori computes the confidence values of the candidate rules by examining them in the top-down order from the lattice.

Think: if the confidence value of $abc \to d$ is below *minconf*, what other candidate rules can be pruned?