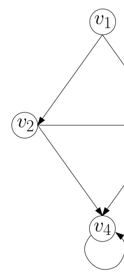
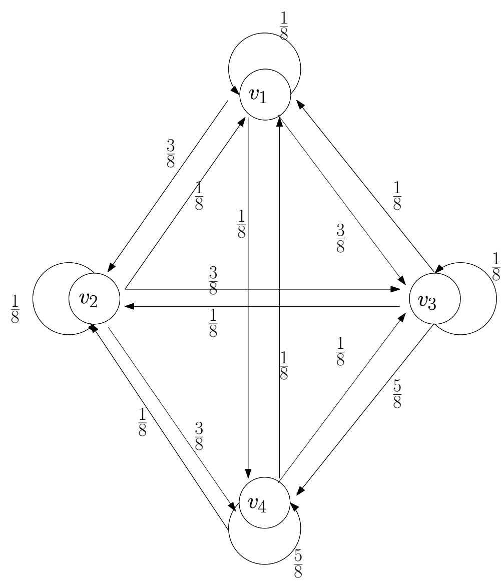
CMSC5724: Exercise List 12

Problem 1. In the following directed graph G, every node represents a webpage, and every edge represents a hyperlink. Consider the "Google random surfing" model with parameter $\alpha = 1/2$. Recall that the model can be regarded as a random walk on a complete graph, where each edge is attached a transition probability. Show this complete graph and give all the transition probabilities.



Solution.



Problem 2. Compute the exact page rank of every node in problem 1.

Solution. Let M be the matrix describing the random walk on the above graph G. We know from the solution of Problem 1:

$$M = \begin{pmatrix} 0.125 & 0.125 & 0.125 & 0.125 \\ 0.375 & 0.125 & 0.125 & 0.125 \\ 0.375 & 0.375 & 0.125 & 0.125 \\ 0.125 & 0.375 & 0.625 & 0.625 \end{pmatrix}$$

It is guaranteed that M has an eigenvalue 1. $(0.125, 0.1563, 0.1953, 0.5234)^T$ is the eigenvector of this eigenvalue satisfying the condition that all the components sum up to 1. Those components are the page ranks of the vertices.

Problem 3. Define r_i as the page rank of v_i in problem 2; and let $P = (r_1, r_2, r_3, r_4)^T$. Use the power method to compute an approximate page rank for every node. Show all the steps of the power method until $Err(t) \leq 0.01$ (see the definition of $Err(t)$ in our lecture notes).

Solution. Let $p(v, t)$ be the approximate page rank of vertex v at t-th round.

Initially, $t = 0$. Set $p(v_1, 0) = 1$, $p(v_2, 0) = p(v_3, 0) = p(v_4, 0) = 0$. In each iteration, use the equation of Slide 8 of our notes to calculate $p(v, t)$ for all vertices v.

Iteration 1. We have

$$\begin{aligned}
p(v_1, 1) &= \frac{1-\alpha}{4} = \frac{1-\frac{1}{2}}{4} = \frac{1}{8} = 0.125 \\
p(v_2, 1) &= \frac{1-\alpha}{4} + \alpha \cdot \frac{p(v_1, 0)}{d^+(v_1)} = \frac{1}{8} + \frac{1}{2} \cdot \frac{1}{2} = 0.375 \\
p(v_3, 1) &= \frac{1-\alpha}{4} + \alpha \left(\frac{p(v_1, 0)}{d^+(v_1)} + \frac{p(v_2, 0)}{d^+(v_2)} \right) = \frac{1}{8} + \frac{1}{2} \cdot (\frac{1}{2} + \frac{0}{2}) = 0.375 \\
p(v_4, 1) &= \frac{1-\alpha}{4} + \alpha \left(\frac{p(v_2, 0)}{d^+(v_2)} + \frac{p(v_3, 0)}{d^+(v_3)} + \frac{p(v_4, 0)}{d^+(v_4)} \right) = \frac{1}{8} + \frac{1}{2} \cdot (\frac{0}{2} + \frac{0}{1} + \frac{0}{1}) = 0.125 \\
Err(1) &= |0.125 - 0.125| + |0.1563 - 0.375| + |0.1953 - 0.375| + |0.5234 - 0.125| \approx 0.7969.
\end{aligned}$$

Iteration 2. Similarly, we get

$$\begin{aligned}
p(v_1, 2) &= \frac{1}{8} = 0.125, \\
p(v_2, 2) &= \frac{1}{8} + \frac{1}{2} \cdot \frac{0.125}{2} \approx 0.1563, \\
p(v_3, 2) &= \frac{1}{8} + \frac{1}{2} \cdot (\frac{0.125}{2} + \frac{0.375}{2}) = 0.25, \\
p(v_4, 2) &= \frac{1}{8} + \frac{1}{2} \cdot (\frac{0.375}{2} + \frac{0.375}{1} + \frac{0.125}{1}) \approx 0.4688. \\
Err(2) &= |0.125 - 0.125| + |0.1563 - 0.1563| + |0.1953 - 0.25| + |0.5234 - 0.4688| \approx 0.1094.
\end{aligned}$$

Iteration 3.

$$\begin{aligned}
p(v_1, 3) &= \frac{1}{8} = 0.125, \\
p(v_2, 3) &= \frac{1}{8} + \frac{1}{2} \cdot \frac{0.125}{2} \approx 0.1563, \\
p(v_3, 3) &= \frac{1}{8} + \frac{1}{2} \cdot (\frac{0.125}{2} + \frac{0.1563}{2}) \approx 0.1953, \\
p(v_4, 3) &= \frac{1}{8} + \frac{1}{2} \cdot (\frac{0.1563}{2} + \frac{0.25}{1} + \frac{0.4688}{1}) \approx 0.5234. \\
Err(3) &= |0.125 - 0.125| + |0.1563 - 0.1563| + |0.1953 - 0.1953| + |0.5234 - 0.5234| \leq 0.01.
\end{aligned}$$

Problem 4. Consider a new definition similar to $Err(t)$:

$$Err'(t) = \max_{i=1}^{n} \left| r_i - P(v_i, t) \right|$$

where the meanings of r_i and $P(v_i, t)$ are the same as in Slide 22 of the lecture notes. Prove that, for any $0 < \epsilon \leq 1$, the power method ensures $Err'(t) \leq \epsilon$ after $t = O(\log \frac{1}{\epsilon})$ rounds.

Solution. In the lecture, we have proved

$$Err(t) \leq \alpha \cdot Err(t-1). \tag{1}$$

Also:

$$Err(0) = \sum_{i=1}^{n} \left| r_i - P(v_i, 0) \right| \leq \sum_{i=1}^{n} (r_i + P(v_i, 0)) \leq \sum_{i=1}^{n} r_i + \sum_{i=1}^{n} P(v_i, 0) = 2. \tag{2}$$

From (1) and (2), we know that $Err(t) \leq \epsilon$ for all

$$t \geq \log_{\frac{1}{\alpha}} \frac{2}{\epsilon};$$

note that $\log_{\frac{1}{\alpha}} \frac{2}{\epsilon} = O(\log \frac{1}{\epsilon})$.

Finally, since

$$Err'(t) \leq Err(t)$$

holds for all $t \geq 0$, we conclude the proof.