DFS and the Proof of White Path Theorem

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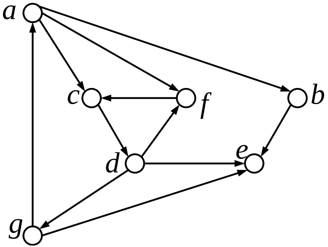
Let's first go over the DFS algorithm through an example.

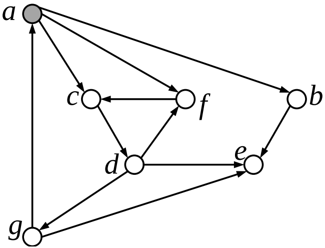
Input



Suppose we start from the vertex a, namely a is the root of DFS tree.

DFS

Firstly, set all the vertices to be white. Then, create a stack S, push the starting vertex a into S and color it gray. Create a DFS Tree with a as the root. We also maintain the time interval $I(u)$ of each vertex u.

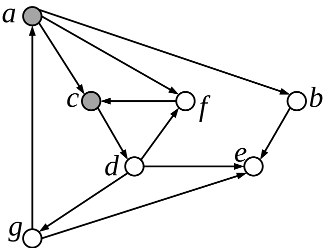


DFS Tree	Time Interval
a	$I(a) = [1, \;]$

$S = (a)$.

DFS

Top of stack: a, which has white out-neighbors b, c, f. Suppose we access c first. Push c into S.



DFS Tree

Time Interval

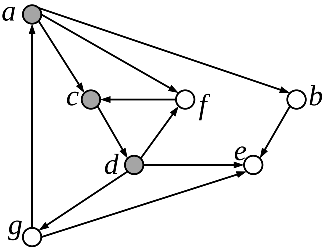
$I(a) = [1, \ ]$

$I(c) = [2, \ ]$

$S = (a, c)$.

DFS

After pushing d into S:

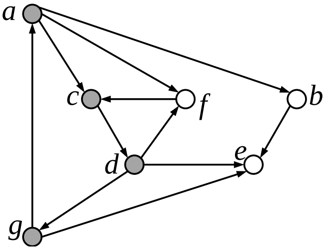


DFS Tree	Time Interval
a	$I(a) = [1, \ ]$
c	$I(c) = [2, \ ]$
d	$I(d) = [3, \ ]$

$S = (a, c, d)$.

DFS

Now d tops the stack. It has white out-neighbors e, f and g. Suppose we visit g first. Push g into S.

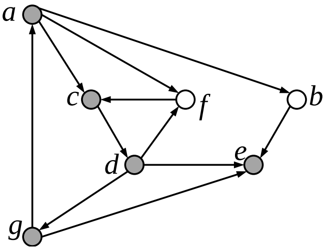


DFS Tree	Time Interval
a	$I(a) = [1, \ ]$
c	$I(c) = [2, \ ]$
d	$I(d) = [3, \ ]$
g	$I(g) = [4, \ ]$

$S = (a, c, d, g)$.

DFS

After pushing e into S:

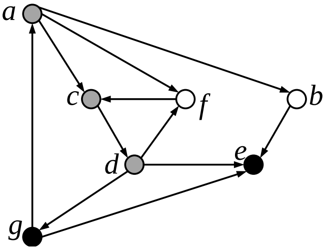


DFS Tree	Time Interval
a	$I(a) = [1, \ ]$
c	$I(c) = [2, \ ]$
d	$I(d) = [3, \ ]$
g	$I(g) = [4, \ ]$
e	$I(e) = [5, \ ]$

$S = (a, c, d, g, e)$.

DFS

e has no white out-neighbors. So pop it from S, and color it black.
Similarly, g has no white out-neighbors. Pop it from S, and color it black.

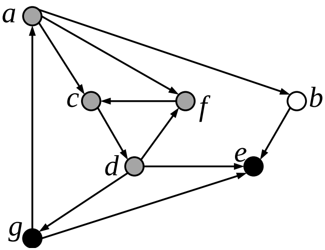


DFS Tree	Time Interval
a	$I(a) = [1, \ ]$
c	$I(c) = [2, \ ]$
d	$I(d) = [3, \ ]$
g	$I(g) = [4, 7]$
e	$I(e) = [5, 6]$

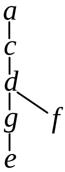
$S = (a, c, d)$.

DFS

Now d tops the stack again. It still has a white out-neighbor f. So, push f into S.



DFS Tree



Time Interval

$I(a) = [1, \ ]$

$I(c) = [2, \ ]$

$I(d) = [3, \ ]$

$I(g) = [4, 7]$

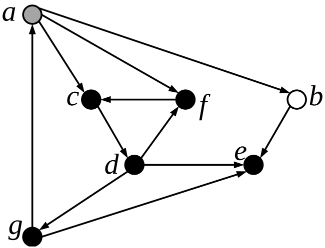
$I(e) = [5, 6]$

$I(f) = [8, \ ]$

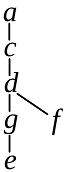
$S = (a, c, d, f)$.

DFS

After popping f, d, c:



DFS Tree



Time Interval

$I(a) = [1,\ ]$

$I(c) = [2, 11]$

$I(d) = [3, 10]$

$I(g) = [4, 7]$

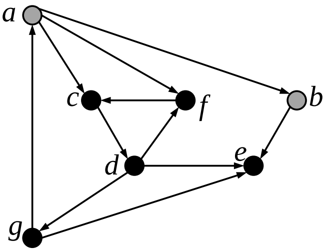
$I(e) = [5, 6]$

$I(f) = [8, 9]$

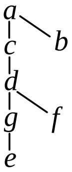
$S = (a)$.

DFS

Now a tops the stack again. It still has a white out-neighbor b. So, push b into S.



DFS Tree



Time Interval

$I(a) = [1, \ ]$

$I(c) = [2, 11]$

$I(d) = [3, 10]$

$I(g) = [4, 7]$

$I(e) = [5, 6]$

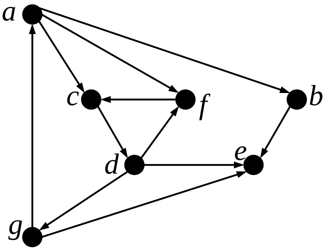
$I(f) = [8, 9]$

$I(b) = [12, \ ]$

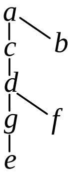
$S = (a, b)$.

DFS

After popping b and a:



DFS Tree



Time Interval

$I(a) = [1, 14]$

$I(c) = [2, 11]$

$I(d) = [3, 10]$

$I(g) = [4, 7]$

$I(e) = [5, 6]$

$I(f) = [8, 9]$

$I(b) = [12, 13]$

$S = ()$.

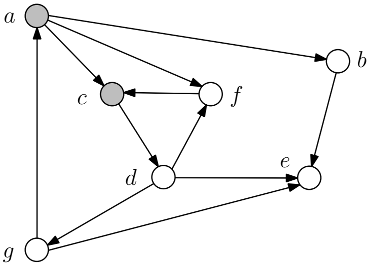
Now, there is no white vertex remaining, our algorithm terminates.

Recall:

White Path Theorem: Let u be a vertex in G. Consider the moment when u is pushed into the stack in the DFS algorithm. Then, a vertex v becomes a proper descendant of u in the DFS-forest **if and only if** the following is true:

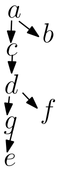
we can go from u to v by travelling only on white vertices.

Example



DFS Tree

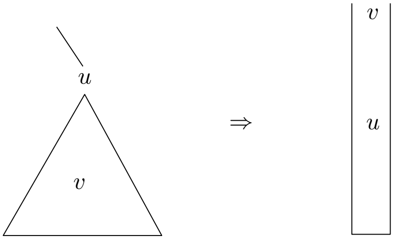
Final DFS Tree



$S =$ | a c |

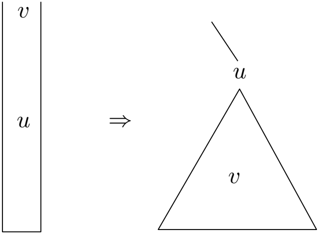
Lemma 1: Consider any two distinct vertices u and v in a DFS-tree. If v is a descendant of u in a DFS-tree, then v enters the stack while u is in the stack.

The proof is left to you.



Lemma 2: Consider any two distinct vertices u and v in a DFS-tree. If v enters the stack while u is in the stack, then v is a descendant of u in a DFS-tree.

The proof is left to you.



Proof of White Path Theorem

White Path Theorem: Let u be a vertex in G. Consider the moment when u is pushed into the stack in the DFS algorithm. Then, a vertex v becomes a proper descendant of u in the DFS-forest **if and only if** the following is true:

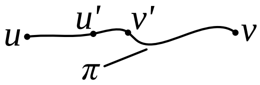
we can go from u to v by travelling only on white vertices.

Proof: The "only-if direction" ($\Rightarrow$): Let v be a descendant of u in the DFS tree. Let π be the path from u to v in the tree. By Lemma 1, all the nodes on π entered the stack after u. Hence, π must be white at the moment when u enters the stack.

Proof of White Path Theorem

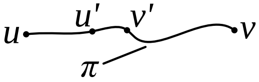
The "if direction" ($\Leftarrow$): When u enters the stack, there is a white path π from u to v. We will prove that all the vertices on π must be descendants of u in the DFS-forest.

Suppose that this is not true. Let v' be the first vertex on π — in the order from u to v — that is not a descendant of u in the DFS-forest. Clearly $v' \neq u$. Let u' be the vertex that precedes v' on π; note that u' is a descendant of u in the DFS-forest.



By Lemma 2, u' entered the stack after u.

Proof of White Path Theorem



Consider the moment when u' turns black (i.e., u' leaving the stack). Node u must remain in the stack currently (first in last out).

1. The color of v' cannot be white.

   Otherwise, v' is a white out-neighbor of u, which contradicts the fact that u' is turning black.

2. Hence, the color of v' must be gray or black.

   Recall that when u entered stack, v' was white. Therefore, v' must have been pushed into the stack while u was still in the stack. By the lemma on Slide 16, v' must be a descendant of u. This, however, contradicts the definition of v'.