Dynamic Programming 5: Optimal BST

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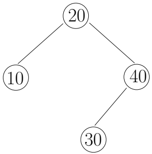
Review: Binary Search Tree (BST)



- Each node stores a **key**.
- The key of an internal node u is larger than any key in the left subtree of u, and smaller than any key in the right subtree of u.

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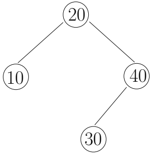


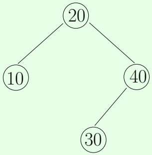
- The **level** of a node u in a BST T — denoted as $level_T(u)$ — equals the number of edges on the path from the root to u.
  - The level of the root is 0.
- The **depth** of a tree is the maximum level of the nodes in the tree.
- Searching for a node u incurs cost proportional to $1 + level_T(u)$.

Let S be a set of n integers. We have learned (from CSCI2100) that a balanced BST on S has depth $O(\log n)$. This is good if all the integers in S are searched with **equal probabilities**.

In practice, not all keys are equally important: some are searched **more often than others**. This gives rise to an interesting question:

> If we know the search frequencies of the integers in S, how to build a better BST to minimize the average search cost?

Example:



Suppose that the search frequencies of 10, 20, 30, and 40 are 40%, 15%, 35%, and 10%, respectively. Then, the average cost of searching for a key in the BST equals:

$$\begin{aligned} & \mathit{freq}(10) \cdot \mathit{cost}(10) + \mathit{freq}(20) \cdot \mathit{cost}(20) + \\ & \mathit{freq}(30) \cdot \mathit{cost}(30) + \mathit{freq}(40) \cdot \mathit{cost}(40) \\ = \; & 40\% \cdot 2 + 15\% \cdot 1 + 35\% \cdot 3 + 10\% \cdot 2 \\ = \; & 2.2. \end{aligned}$$

The Optimal BST Problem

Input:

- A set S of n integers: $\{1, 2, ..., n\}$;
- An array W where $W[i]$ $(1 \leq i \leq n)$ stores a positive integer weight.

Output: A BST T on S with the smallest **average cost**

$$avgcost(T) = \sum_{i=1}^{n} W[i] \cdot cost_T(i).$$

where $cost_T(i) = 1 + level_T(i)$ is the number of nodes accessed to find the key i in T.

We will solve a more general version of the problem.

Input:

- S and W same as before;
- Integers a, b satisfying $1 \leq a \leq b \leq n$.

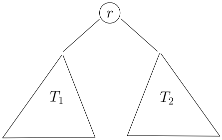
Output: A BST T on $\{a, a+1, ..., b\}$ with the smallest **average cost**:

$$avgcost(T) = \sum_{i=a}^{b} W[i] \cdot cost_T(i).$$

Fact: The root of T must have a key $r \in [a, b]$.

After the root key r is fixed, we know:

- the root's left subtree is a BST T_1 on $S_1 = \{a, ..., r-1\}$, and
- the root's right subtree is a BST T_2 on $S_2 = \{r+1, ..., b\}$.



Lemma: Let T, T_1, and T_2 be defined as above. Then:

$$avgcost(T) = \left(\sum_{i=a}^{b} W[i]\right) + avgcost(T_1) + avgcost(T_2).$$

Proof:

$$
\begin{aligned}
& avgcost(T) \\
= \; & \sum_{i=a}^{b} W[i] \cdot cost_T(i) = \sum_{i=a}^{b} W[i] \cdot (1 + level_T(i)) \\
= \; & \left(\sum_{i=a}^{b} W[i] \right) + \sum_{i=a}^{b} W[i] \cdot level_T(i) \\
= \; & \left(\sum_{i=a}^{b} W[i] \right) + \left(\sum_{i=a}^{r-1} W[i] \cdot level_T(i) \right) + \left(\sum_{i=r+1}^{b} W[i] \cdot level_T(i) \right)
\end{aligned}
$$

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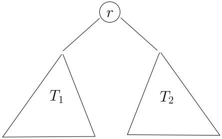
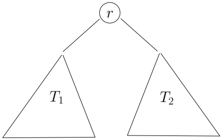


$$
\begin{aligned}
= & \left(\sum_{i=a}^{b} W[i]\right) + \\
& \left(\sum_{i=a}^{r-1} W[i] \cdot (1 + level_{T_1}(i))\right) + \left(\sum_{i=r+1}^{b} W[i] \cdot (1 + level_{T_2}(i))\right) \\
= & \left(\sum_{i=a}^{b} W[i]\right) + \left(\sum_{i=a}^{r-1} W[i] \cdot cost_{T_1}(i)\right) + \left(\sum_{i=r+1}^{b} W[i] \cdot cost_{T_2}(i)\right) \\
= & \left(\sum_{i=a}^{b} W[i]\right) + avgcost(T_1) + avgcost(T_2).
\end{aligned}
$$



Define *optavg*(a, b) as

- 0, if $a > b$;
- the smallest average cost of a BST on $\{a, a+1, ..., b\}$, otherwise.

Define *optavg*$(a, b \mid r)$ as the optimal average cost of a BST, **on condition that** the BST has $r \in [a, b]$ as the key of the root.

By the previous lemma, we have:

$$\begin{aligned} & optavg(a, b \mid r) \\ = & \left(\sum_{i=a}^{b} W[i]\right) + optavg(a, r-1) + optavg(r+1, b). \end{aligned}$$

Example: $S = \{1, 2, 3, 4\}$; $W = (40, 15, 35, 10)$.

Consider choosing 2 as the root key.

$$\begin{aligned} & optavg(1, 4 \mid 2) \\ = \; & \left(\sum_{i=1}^{4} W[i]\right) + optavg(1, 1) + optavg(3, 4) \\ = \; & 100 + 40 + 55 = 195. \end{aligned}$$

Hence, **among all BSTs with root key 2**, the best BST has average cost 195.

The **recursive structure** of the problem:

$$
\begin{aligned}
& optavg(a, b) \\
= \; & \min_{r=a}^{b} optavg(a, b \mid r) \\
= \; & \left(\sum_{i=a}^{b} W[i]\right) + \min_{r=a}^{b} \left\{ optavg(a, r-1) + optavg(r+1, b) \right\}.
\end{aligned}
$$

With dynamic programming, we can compute $optavg(1, n)$ in $O(n^3)$ time (left as a special exercise).

Strictly speaking, we have not produced the optimal BST yet. However, fixing the issue should be fairly standard to you at this moment: the piggyback technique allows you to build the tree in the same time complexity as computing $opt(1, n)$. This is left as a special exercise.