Divide and Conquer

Yufei Tao

Department of Computer Science and Engineering
Chinese University of Hong Kong

In this lecture, we will discuss the **divide and conquer** technique for designing algorithms with strong performance guarantees. Our discussion will be based on the following problems:

1. Sorting (a review of merge sort)
2. Counting inversions
3. Dominance counting
4. Matrix multiplication

Principle of divide and conquer:

> Divide a problem into sub-problems, solve the sub-problems by recursion, and derive the final answer from the sub-problems' outputs.

Sorting

Sorting

Problem: Given an array A of n distinct integers, produce another array where the same integers have been arranged in ascending order.

- **Divide:** Let A_1 the array containing the first $\lceil n/2 \rceil$ elements of A, and A_2 be the array containing the other elements of A.
  Sort A_1 and A_2 recursively.
- **Conquer:** Merge the two sorted arrays A_1 and A_2 in ascending order. This can be done in $O(n)$ time.

This is the merge sort algorithm.

Sorting

Running Time: Let $f(n)$ denote the worst-case cost of the algorithm on an array of size n. Then:

$$f(n) \leq 2 \cdot f(\lceil n/2 \rceil) + O(n)$$

which gives $f(n) = O(n \log n)$.

Counting Inversions

Counting Inversions

Let: A = an array of n distinct integers.

An **inversion** is a pair of (i, j) such that

- $1 \leq i < j \leq n$, and
- $A[i] > A[j]$.

Example: Consider $A = (10, 3, 9, 8, 2, 5, 4, 1, 7, 6)$.
Then $(1, 2)$ is an inversion because $A[1] = 10 > A[2] = 3$. So are $(1, 3), (3, 4), (4, 5)$, and so on.
There are in total 29 inversions.

Think: How many inversions can there be in the worst case?
Answer: $\binom{n}{2} = \Theta(n^2)$.

Counting Inversions

Problem: Given an array A of n distinct integers, count the number of inversions.

We will do in the class: $O(n \log^2 n)$ time.
You will do as an exercise: $O(n \log n)$ time.

Counting Inversions

- **Divide:** Let A_1 the array containing the first $\lceil n/2 \rceil$ elements of A, and A_2 be the array containing the other elements of A.

  Solve the "counting inversions" problem recursively on A_1 and A_2, respectively. By doing so, we have already obtained the number m_1 of inversions in A_1, and similarly, the number m_2 for A_2.

- **Conquer:**
  It remains to count the number of **crossing inversions** (i, j) where $i \in A_1$ and $j \in A_2$.

Counting Inversions

A_1 = the array containing the first $\lceil n/2 \rceil$ elements of A
A_2 = the array containing the other elements of A.

Sort A_1.
For each element $e \in A_2$, count how many crossing inversions e produces using **binary search**.

Example (cont.): $A = (10, 3, 9, 8, 2, 5, 4, 1, 7, 6)$.
$A_1 = (2, 3, 8, 9, 10)$ (sorted), $A_2 = (5, 4, 1, 7, 6)$

Element 5 produces 3 crossing inversion
Element 4 produces 3, too.
Elements 1, 7, and 6 produce 5, 3, and 3 crossing inversions, respectively.

- **Think:** How to obtain each count with binary search?

In total, $n/2$ binary searches are performed, which takes $O(n \log n)$ time.

Counting Inversions

Running Time: Let $f(n)$ denote the worst-case cost of the algorithm on an array of size n. Then:

$$f(n) \leq 2 \cdot f(\lceil n/2 \rceil) + O(n \log n)$$

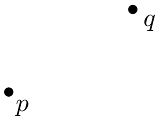
which gives $f(n) = O(n \log^2 n)$.

Dominance Counting

Dominance Counting

Denote by $\mathbb{Z}$ the set of integers.
Given a point p in two-dimensional space $\mathbb{Z}^2$, denote by $p[1]$ and $p[2]$ its x- and y-coordinate, respectively.

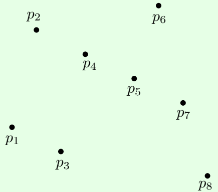
Given two distinct points p and q, we say that q **dominates** p if $p[1] \leq q[1]$ and $p[2] \leq q[2]$; see the figure below:



Dominance Counting

Let P be a set of n points in $\mathbb{Z}^2$ with distinct x-coordinates. Find, for each point $p \in P$, the number of points in P that are dominated by p.

Example:



We should output: $(p_1, 0), (p_2, 1), (p_3, 0), (p_4, 2), (p_5, 2), (p_6, 5), (p_7, 2), (p_8, 0)$.

Dominance Counting

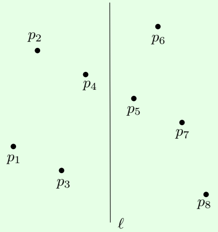
Let P be a set of n points in $\mathbb{Z}^2$ with distinct x-coordinates. Find, for each point $p \in P$, the number of points in P that are dominated by p.

We will do in the class: $O(n \log^2 n)$ time.
You will do as an exercise: $O(n \log n)$ time.

Dominance Counting

Divide: Find a vertical line ℓ such that P has $\lceil n/2 \rceil$ points on each side of the line.

Example:



Think: How to find such ℓ in $O(n \log n)$ time? How about $O(n)$ time?

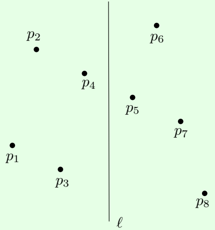
Dominance Counting

Divide:

P_1 = the set of points of P on the left of ℓ

P_2 = the set of points of P on the right of ℓ

Example:



$$P_1 = \{p_1, p_2, p_3, p_4\}$$

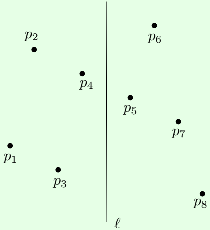
$$P_2 = \{p_5, p_6, p_7, p_8\}.$$

Dominance Counting

Divide:

Solve the dominance counting problem on P_1 and P_2 separately.

Example:



On P_1, we have obtained: $(p_1, 0), (p_2, 1), (p_3, 0), (p_4, 2)$.

On P_2, we have obtained: $(p_5, 0), (p_6, 1), (p_7, 0), (p_8, 0)$.

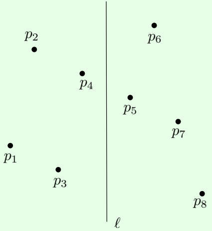
The counts obtained for the points in P_1 are final (**think:** why?).

Dominance Counting

Conquer:

It remains to count, for each point $p_2 \in P_2$, how many points in P_1 it dominates.

Example:



On P_2, we have obtained: $(p_5, 0), (p_6, 1), (p_7, 0), (p_8, 0)$.

Regarding p_5, for example, we still need to find out that it dominates 2 points from P_1.

The x-coordinates do not matter any more!

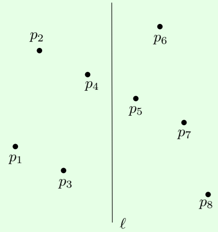
Dominance Counting

Conquer:

Sort P_1 by **y-coordinate**.

Then, for each point $p_2 \in P_2$, we can obtain the number points in P_1 dominated by p_2 using binary search.

Example:



P_1 in ascending of y-coordinate: p_3, p_1, p_4, p_2.

How to perform binary search to obtain the fact that p_5 dominates 2 points in P_1?

- Search using the y-coordinate of p_5.

Dominance Counting

Analysis:

Let $f(n)$ be the worst-case running time of the algorithm on n points. Then:

$$f(n) \leq 2f(\lceil n/2 \rceil) + O(n \log n)$$

which solves to $f(n) = O(n \log^2 n)$.

Matrix Multiplication

Matrix Multiplication

Problem: Given two $n \times n$ matrices A and B, compute their product AB.

We store an $n \times n$ matrix with an array of length n^2 in "row-major" order.

Example: $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is stored as $(1, 2, 3, 4)$.

Note that any $A[i, j]$ — the element of A at the i-th row and j-th column — can be accessed in $O(1)$ time.

Trivial: $O(n^3)$ time
We will do in the class: $O(n^{2.81})$ time for n being a power of 2
You will do as an exercise: $O(n^{2.81})$ time for any n.

Matrix Multiplication

Warm Up: Suppose we want to compute $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix}$. How many multiplication operations do we need to perform?

Trivial: 8.

Non-trivial: 7.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} p_5 + p_4 - p_2 + p_6 & p_1 + p_2 \\ p_3 + p_4 & p_1 + p_5 - p_3 - p_7 \end{bmatrix}$$

where

$$\begin{aligned} p_1 &= a(f - h) \\ p_2 &= (a + b)h \\ p_3 &= (c + d)e \\ p_4 &= d(g - e) \\ p_5 &= (a + d)(e + h) \\ p_6 &= (b - d)(g + h) \\ p_7 &= (a - c)(e + f) \end{aligned}$$

Matrix Multiplication (Strassen's Algorithm)

Recall that the input A and B are order-n (i.e., $n \times n$) matrices. Assume for simplicity that n is a power of 2. Divide each of A and B into 4 submatrices of order $n/2$:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

It is easy to verify:

$$AB = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{bmatrix}$$

How many order-$(n/2)$ matrix multiplications do we need?

Trivial: 8.
Non-trivial: 7 — see the next slide.

Matrix Multiplication

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} p_5 + p_4 - p_2 + p_6 & p_1 + p_2 \\ p_3 + p_4 & p_1 + p_5 - p_3 - p_7 \end{bmatrix}$$

$$\begin{aligned}
p_1 &= A_{11}(B_{12} - B_{22}) \\
p_2 &= (A_{11} + A_{12})B_{22} \\
p_3 &= (A_{21} + A_{22})B_{11} \\
p_4 &= A_{22}(B_{21} - B_{11}) \\
p_5 &= (A_{11} + A_{22})(B_{11} + B_{22}) \\
p_6 &= (A_{12} - A_{22})(B_{21} + B_{22}) \\
p_7 &= (A_{11} - A_{21})(B_{11} + B_{12})
\end{aligned}$$

If $f(n)$ is the worst-case time of computing the product of two order-n matrices, then each of p_i $(1 \le i \le 7)$ can be computed in $f(n/2) + O(n^2)$ time.

Matrix Multiplication

Therefore:

$$f(n) \leq 7f(n/2) + O(n^2)$$

which solves to $f(n) = O(n^{\log_2 7}) = O(n^{2.81})$.