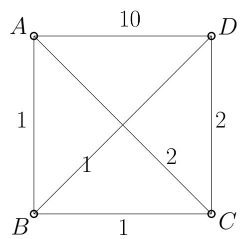
CSCI3160: Regular Exercise Set 11

Prepared by Yufei Tao

Problem 1. The figure below shows a weighted simple graph $G = (V, E)$, where the integers indicate the edge weights.



Use the graph to explain why our approximation algorithm for TSP (the traveling salesman problem) can no longer ensure an approximation ratio of 2 without triangle inequality.

Problem 2*. Give an input to show that our approximation TSP algorithm does not guarantee an approximation ratio of 1.6.

Problem 3. Let $G = (V, E)$ be a simple undirected graph where each edge $e \in E$ is assigned a non-negative weight $w(e)$ (note: G may not be a complete graph). G is connected. A *spanning walk* of G is a walk that visits every vertex at least once (the walk may travel on the same edge multiple times). Let OPT_G be the shortest length of all spanning walks. Design a poly($|V|$)-time algorithm to find a spanning walk with length at most $2 \cdot \text{OPT}_G$.

Problem 4 (No Triangle Inequality No Approximation). In this problem, you will see that if once the triangle inequality requirement is dropped, then it is not possible to guarantee any constant approximation ratio for the TSP problem in polynomial time unless P = NP.

Let us restate the TSP problem without triangle inequality. Let $G = (V, E)$ be a simple undirected complete graph where each edge $e \in E$ is associated with a non-negative weight $w(e)$. A Hamiltonian cycle (as before) is a cycle that includes all the vertices of V. The goal is the find a Hamiltonian cycle with the smallest length OPT, defined as the total weight of the edges on the cycle. A ρ-approximate algorithm is required to find a Hamiltonian cycle whose length is at most $\rho \cdot \text{OPT}$.

We will utilize a fact: the Hamiltonian-cycle problem, as defined, is NP-hard.

> **The Hamiltonian Cycle Problem:** Given a simple undirected graph G, decide whether G contains a Hamiltonian cycle.

The fact (whose proof is not required in this course) indicates that, unless P = NP, no polynomial algorithm exist for detecting the existence of a Hamiltonian cycle.

Now, suppose that you are given an algorithm $\mathcal{A}$ for TSP that claims to guarantee an approximation ratio ρ in polynomial time, for some constant $\rho \geq 1$. Next, we will use $\mathcal{A}$ to solve the Hamiltonian cycle problem in polynomial time. Given an input $G = (V, E)$ to the Hamiltonian cycle problem, we construct a complete graph $G' = (V', E')$ as follows:

- $V' = V$.

- For any edge $e = \{u, v\} \in E'$, set the weight of e to 1 if e exists in G; otherwise, set the weight to $\rho|V| + 1$.

Run $\mathcal{A}$ on G' to find a Hamiltonian cycle C in polynomial time. Prove: G (the original graph) has any Hamiltonian cycle if and only if the length of C is $|V|$.