CSCI3160: Regular Exercise Set 11

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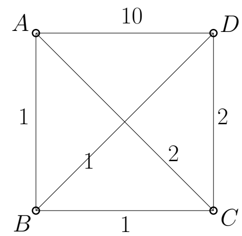
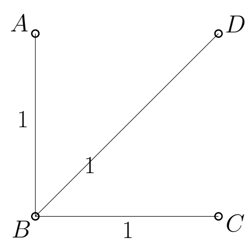
Problem 1. The figure below shows a weighted simple graph $G = (V, E)$, where the integers indicate the edge weights.



Use the graph to explain why our approximation algorithm for TSP (the traveling salesman problem) can no longer ensure an approximation ratio of 2 without triangle inequality.

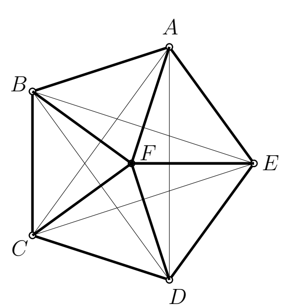
Solution. The figure below is an MST of G.



$ABCBDBA$ is a closed walk of this MST. Given this closed walk, our algorithm will produce a Hamiltonian cycle $A \to B \to C \to D \to A$, whose length is 14. An optimal Hamiltonian cycle, however, should have a length of 6 ($A \to B \to D \to C \to A$).

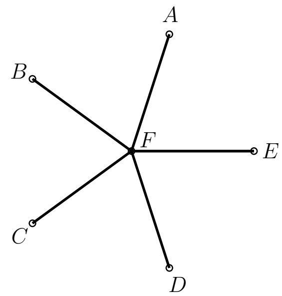
Problem 2*. Give an input to show that our approximation TSP algorithm does not guarantee an approximation ratio of 1.6.

Solution. Consider the following graph G with 6 vertices:



Every thick edge has weight 1, whereas every thin edge has weight 2. Note that the triangle inequality is satisfied. Clearly, an optimal Hamiltonian cycle has length 6, as can be achieved by the cycle $A \to B \to C \to D \to E \to F \to A$.

Suppose that our algorithm finds the following MST:



$FAFCFEFBFDF$ is a closed walk of this MST. Given this closed walk, our algorithm will produce a Hamiltonian cycle $F \to A \to C \to E \to B \to D \to F$, whose length is 10. The approximation ratio is $10/6 > 1.6$.

Problem 3. Let $G = (V, E)$ be a simple undirected graph where each edge $e \in E$ is assigned a non-negative weight $w(e)$ (note: G may not be a complete graph). G is connected. A *spanning walk* of G is a walk that visits every vertex at least once (the walk may travel on the same edge multiple times). Let OPT_G be the shortest length of all spanning walks. Design a poly($|V|$)-time algorithm to find a spanning walk with length at most $2 \cdot \mathrm{OPT}_G$.

Solution. Let λ be the weight sum of all the edges in a minimum spanning tree of G. Any spanning walk must contain the edges of some spanning tree. Therefore, $\mathrm{OPT}_G \geq \lambda$. On the other hand, our approximation algorithm for the traveling salesman problem (discussed in the lecture) finds a spanning walk with length at most $2 \cdot \lambda$, which is thus at most $2 \cdot \mathrm{OPT}_G$.

Problem 4 (No Triangle Inequality No Approximation). In this problem, you will see that if the triangle inequality requirement is dropped, then it is not possible to guarantee any constant approximation ratio for the TSP problem in polynomial time unless P = NP.

Let us restate the TSP problem without triangle inequality. Let $G = (V, E)$ be a simple undirected complete graph where each edge $e \in E$ is associated with a non-negative weight $w(e)$. A Hamiltonian cycle (as before) is a cycle that includes all the vertices of V. The goal is the find a Hamiltonian cycle with the smallest length OPT, defined as the total weight of the edges on the cycle. A ρ-approximate algorithm is required to find a Hamiltonian cycle whose length is at most $\rho \cdot \mathrm{OPT}$.

We will utilize a fact: the Hamiltonian-cycle problem, as defined, is NP-hard.

> **The Hamiltonian Cycle Problem:** Given a simple undirected graph G (which may not be a complete graph), decide whether G contains a Hamiltonian cycle.

The fact (whose proof is not required in this course) indicates that, unless P = NP, no polynomial algorithm exist for detecting the existence of a Hamiltonian cycle.

Now, suppose that you are given an algorithm $\mathcal{A}$ for TSP that claims to guarantee an approximation ratio ρ in polynomial time, for some constant $\rho \geq 1$. Next, we will use $\mathcal{A}$ to solve the Hamiltonian cycle problem in polynomial time. Given an input $G = (V, E)$ to the Hamiltonian cycle problem, we construct a complete graph $G' = (V', E')$ as follows:

- $V' = V$.
- For any edge $e = \{u, v\} \in E'$, set the weight of e to 1 if e exists in G; otherwise, set the weight to $\rho|V| + 1$.

Run $\mathcal{A}$ on G' to find a Hamiltonian cycle C in polynomial time. Prove: G (the original graph) has any Hamiltonian cycle if and only if the length of C is $|V|$.

Solution. We will first prove the if-direction, namely, C having length $|V|$ implies that G has a Hamiltonian cycle. Note that C has exactly $|V|$ edges. It thus follows that every edge of C must have weight 1 and, hence, exists in G. This means that C itself is a Hamiltonian cycle in G.

Next, we prove the only-if direction, namely, G having a Hamiltonian cycle implies that C must have length $|V|$. As G has a Hamiltonian cycle, this cycle must also exist in G'. Therefore, the TSP problem on G' has an optimal solution whose length $\text{OPT} = |V|$. We claim that C cannot use any edge $e \in E'$ that does not exist in G. If so, then the length of C must be at least $(|V| - 1) + (\rho|V| + 1) = \rho|V| + |V|$, which is strictly larger than $\rho|V| = \rho \cdot \text{OPT}$. This contradicts the fact that $\mathcal{A}$ is ρ-approximate.