Propagation of Uncertainty within the Affine Transformation Application on Contours

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Abstract

This paper aims to investigate the propagation of horizontal errors, which are included in a digital contour file of a topographic map by applying the affine transformation of coordinates to this file. The application of the process increases existing horizontal errors, and therefore, these errors may have a direct proportionate impact on the elevation errors, which due to geometry, they co-exist within the digital file. This paper investigates the increase of these horizontal errors, as well as their effect on the respective elevation errors. The investigation applies to various usage cases of affine transformation of coordinates.

Keywords

affine transformation, error propagation, horizontal errors, elevation errors

I. INTRODUCTION

A basic process in cartographic information management is the Geometric Transformation of Coordinates (GTC) or Georeference, as it is otherwise known (Govil et al., 2005; Habbib et al., 2004; Nakos, 1990; Robinson et al., 1984; Sprinsky; Willneff et al., 2006; Yakamura et al., 2004).

During this process, a set of cartographic information is subject to change of reference system, which means that this set is "distorted" so as to adapt to the new reference system of interest.

This transformation is effected by means of a resultant application process of a model (Maling 1991), which is analysed in three influence components in the x and y axes (Doytsher et al., 1984, Nakos, 1990; Paraschakis et al., 1988; Tournas, 1994):

- Translation (in *X*, in *Y*)
- Rotation (in *X*, in *Y*)
- Scale (in *X*, in *Y*)

The models used are either analytical or linear (or grid-to-grid models, as also called), or polynomial (Chiu et al., 2003; Maling, 1991). In this paper, we examine a largely used linear model, i.e. the affine transformation, and we investigate the error propagation, which is produced by the affine transformation in the process output (Achilleos, 2002; Crain et al., 1993;Dorozhyiskiy et al., 1997; Gong, 1992; Gong et al., 1995; Gong et al., 2000).

II. AFFINE TRANSFORMATION OF COORDINATES

The Affine Transformation of Coordinates (A.T.C.) is a two-dimensional linear function (linear model) of the type (Nakos, 1990):

$$x = a_1 + a_2 \cdot X + a_3 \cdot Y \qquad (1)$$
$$y = a_4 + a_5 \cdot X + a_6 \cdot Y$$

where:

X, Y: plane coordinates prior to the transformation

x, y : plane coordinates after the transformation

a_i :transformation coefficients ($i = 1, 2, ..., 6$)

The main advantage of GTC linear models is that there is no need to know the type and the parameters of the map projection used (Kraus et al., 2007; Leyk et al., 2005; Zheng et al., 1997). This flexibility is very useful in photographic data cases or in the use of multiple map sheets for covering the area under examination (Chen et al., 2006; Liang et al., 1995; Noguchi et al., 2004). However, we should know the coordinates of common points in both map projections, with a relative high accuracy. Another advantage is the simplicity of the functions used, the simplicity of operations and the speed of the transformation performance.

The a_i coefficients may be combined and give the distortions and translations that are included in the transformation as follows (Table 1) (Nakos, 1990; Nakos et al., 1994):

Table 1. Distortions and translations produced by the Transformation in connection with the a_i coefficients

	in *X*	in *Y*
Translation	a_1	a_4
Rotation	$\arctan(-a_5/a_2)$	$\arctan(a_3/a_6)$
Scaling	$SQR(a_2^2 + a_5^2)$	$SQR(a_3^2 + a_6^2)$

Before applying the affine transformation, it is suggested to follow certain steps for the data preparation. Firstly, a parallel translation of each of the two reference systems is effected relative to the center of gravity of the coordinates of the area under examination (Figure 1). Then one reference system is rotated towards the other (Figure 2) and, in the end a scale change is effected so that both reference systems are identified as much as possible (Figure 3). The rotation and the scale

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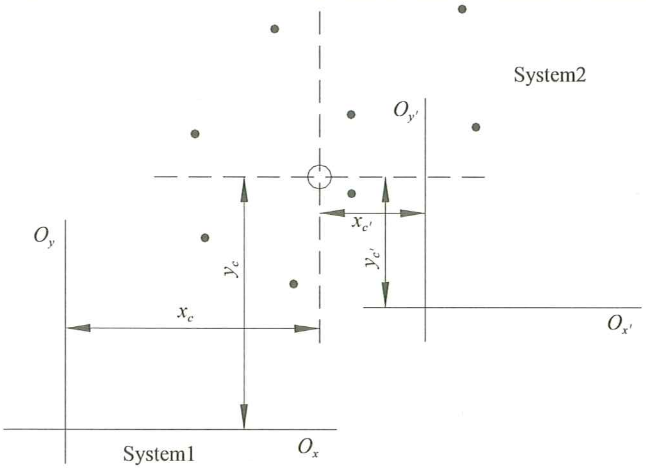

Figure 1. Parallel translation of the reference systems relative to their centers of gravity

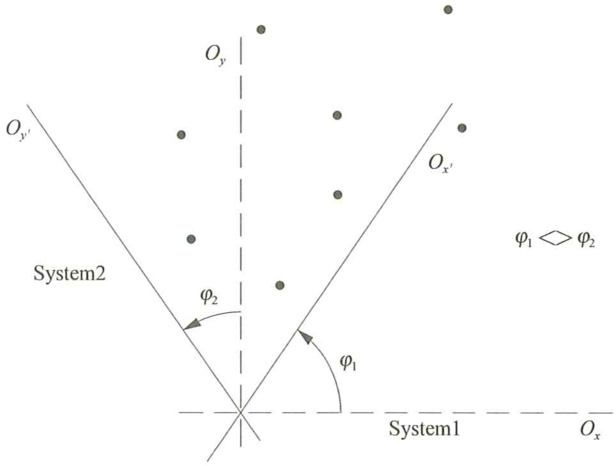

Figure 2. Rotation of the reference system in the affine transformation

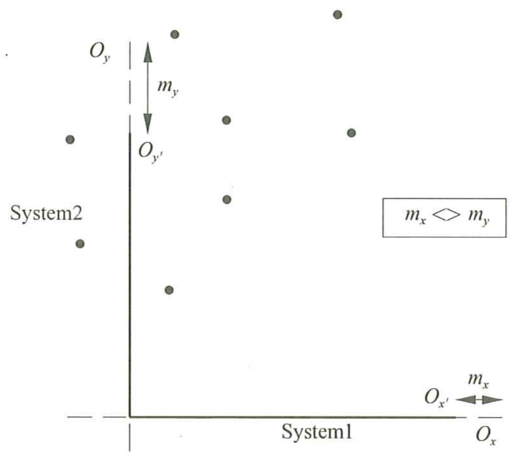

Figure 3. Scale change of the reference system in the affine transformation

change are uniform in the two reference axes of the system. The reason for applying these steps prior to the affine transformation is due to the model's mathematical structure, which maximizes errors σ_x and σ_y when the coordinates of the points transformed are maximized, or when the a_i coefficients are maximized. The phenomenon of error maximization is fictitious, and if the center of gravity, rotation and scale are not previously changed, then errors σ_x and σ_y will be very large in size, which in fact is not valid (Achilleos, 2002, Mikhail et al., 2006; Nakos, 1990; Paraschakis et al., 1988).

The parallel translation relative to the center of gravity is effected by the following operations:

$$X_i = X_i - x_c$$
$$Y_i = Y_i - y_c \qquad (2)$$

where: $$x_c = \frac{\sum_{i=1}^{n} X_i}{n}, \; y_c = \frac{\sum_{i=1}^{n} Y_i}{n}$$

The application of the rotation is carried out by calculating the mean rotation presented by the points, which are used as the control points in the transformation:

$$\phi = E[\phi_i] \qquad (3)$$

A similar logic is followed in scale change, which is carried out by calculating the mean scale, as follows:

$$m = E[l_i / l_i'] \qquad (4)$$

Following this, after the transformation is performed, the coordinates return in their absolute position, by adding again the center of gravity of the second reference system:

$$x_i = x_i + x_c$$
$$y_i = y_i + y_c \qquad (5)$$

The relative distribution of points, the coordinates of which are transformed, is not affected by the translation, rotation and enlargement (or diminution) and, therefore, the figure created by these points is not affected.

III. THE EFFECT OF THE GTC ON DATA ACCURACY

Each analysis processing and form that a data group can receive produces errors in the data and reduces the reliability of the results. In the GTC this is due to various factors, such as the degree of adaptation of the model used, the rounding-up in mathematical operations, the accuracy of data which is included in the processing and analysis, etc.

This is not the case in GTC analytical models, because they rely on analytic functions, which are pre-defined and independent from any applications of coordinate geometric transformation.

Selecting the most appropriate GTC model and ensuring good data accuracy give the optimum result with regard to the model's adaptation and the reliability-accuracy of the results.

There are two basic elements, which describe the accuracy and reliability of a model to be used in GTC. These are (Nakos, 1990):

- Variance–Covariance table of the model's solving–adaptation to the transformation data.
- Aposteriori standard deviation of the model's solving-

adaptation to the transformation data.

The aposteriori standard deviation is the size which marks the entire quality of the model's solving-adaptation to the transformation data, while the variance–covariance table gives the uncertainties of the coefficients/parameters of the transformation model and describes the partial quality of the solving-adaptation.

A great number of aposteriori standard deviation in combination with a small number of data refers to the increase of the data size and the improvement of their quality, in order to utilize the transformation model, which is solved.

The variance-covariance table may provide elements for the three components of the transformation (translation, rotation, scale) and indicate the ones that suffer in the particular transformation.

A. The effect of linear and polynomial GTC models on the accuracy of the data horizontal position

The application of a linear or polynomial GTC model to the coordinates (X, Y) of a sum of points gives new coordinates (x, y). The accuracy of the transformed coordinates (x, y) is described with the following components:

$$\sigma_x, \sigma_y, \sigma_{xy} \quad (6)$$

These components are defined by applying the error propagation law (7, 8):

VARIANCE:

$$f(x) = f(x_1, x_2, \cdots, x_n)$$

$$\sigma^2 f(x) = \sum_{i=1}^{n}\left(\frac{\partial f}{\partial x_i}\right)^2 \cdot \sigma_{x_i}^2 + 2\cdot\sum_{i=1}^{n}\sum_{j=i+1}^{n}\left(\frac{\partial f}{\partial x_i}\cdot\frac{\partial f}{\partial x_j}\right)\cdot\sigma_{x_i x_j} \quad (7)$$

COVARIANCE:

$$f_1(x) = f_1(x_1, x_2, \cdots, x_n)$$

$$f_2(x) = f_2(x_1, x_2, \cdots, x_n)$$

$$\sigma_{f_1(x)f_2(x)} = \sum_{i=1}^{n}\left(\frac{\partial f_1}{\partial x_i}\cdot\frac{\partial f_2}{\partial x_i}\right)\cdot\sigma_{x_i}^2 + \sum_{i=1}^{n}\sum_{j=i+1}^{n}\left(\frac{\partial f_1}{\partial x_i}\cdot\frac{\partial f_2}{\partial x_j} + \frac{\partial f_1}{\partial x_j}\cdot\frac{\partial f_2}{\partial x_i}\right)\cdot\sigma_{x_i x_j} \quad (8)$$

Relations (7, 8) show that the accuracy of the transformed coordinates (x,y) depends on the following factors:

- Coordinates prior to the application of the transformation and their variances-covariances X, Y, σ_X, σ_Y, σ_{XY}
- Parameters/coefficients of the transformation model and their variances-covariances a_i, σ_{ai}, σ_{aiaj}

B. Propagation of the position error due to the affine transformation

The function of the Affine Transformation of Coordinates (ATC) (1) and the application of the error propagation law (7, 8) give the following results (9, 10):

$$\begin{aligned}\sigma_x^2 &= a_2^2\cdot\sigma_X^2 + a_3^2\cdot\sigma_Y^2 + \sigma_{a_1}^2 + X^2\cdot\sigma_{a_2}^2 + Y^2\cdot\sigma_{a_3}^2 + \\ &2\cdot a_2\cdot a_3\cdot\sigma_{XY} + 2\cdot a_2\cdot\sigma_{Xa_1} + 2\cdot a_2\cdot X\cdot\sigma_{Xa_2} + \\ &2\cdot a_2\cdot Y\cdot\sigma_{Xa_3} + 2\cdot a_3\cdot\sigma_{Ya_1} + 2\cdot a_3\cdot X\cdot\sigma_{Ya_2} + \\ &2\cdot a_3\cdot Y\cdot\sigma_{Ya_3} + 2\cdot X\cdot\sigma_{a_1a_2} + 2\cdot Y\cdot\sigma_{a_1a_3} + 2\cdot X\cdot Y\cdot\sigma_{a_2a_3} \\ \sigma_y^2 &= a_5^2\cdot\sigma_X^2 + a_6^2\cdot\sigma_Y^2 + \sigma_{a_4}^2 + X^2\cdot\sigma_{a_5}^2 + Y^2\cdot\sigma_{a_6}^2 + \\ &2\cdot a_5\cdot a_6\cdot\sigma_{XY} + 2\cdot a_5\cdot\sigma_{Xa_1} + 2\cdot a_5\cdot X\cdot\sigma_{Xa_5} + \\ &2\cdot a_5\cdot Y\cdot\sigma_{Xa_6} + 2\cdot a_6\cdot\sigma_{Ya_4} + 2\cdot a_6\cdot X\cdot\sigma_{Ya_5} + \\ &2\cdot a_6\cdot Y\cdot\sigma_{Ya_6} + 2\cdot X\cdot\sigma_{a_4a_5} + 2\cdot Y\cdot\sigma_{a_4a_6} + 2\cdot X\cdot Y\cdot\sigma_{a_5a_6}\end{aligned} \quad (9)$$

$$\begin{aligned}\sigma_{xy} &= a_2\cdot a_5\cdot\sigma_X^2 + a_3\cdot a_6\cdot\sigma_Y^2 + (a_2\cdot a_6 + a_3\cdot a_5)\cdot\sigma_{XY} + \\ &a_5\cdot\sigma_{Xa_1} + a_5\cdot X\cdot\sigma_{Xa_2} + a_5\cdot Y\cdot\sigma_{Xa_3} + a_2\cdot\sigma_{Xa_4} + \\ &a_2\cdot X\cdot\sigma_{Xa_5} + a_2\cdot Y\cdot\sigma_{Ya_6} + a_6\cdot\sigma_{Ya_1} + a_6\cdot X\cdot\sigma_{Ya_2} + \\ &a_6\cdot Y\cdot\sigma_{Ya_3} + a_3\cdot\sigma_{Ya_4} + a_3\cdot X\cdot\sigma_{Ya_5} + a_3\cdot Y\cdot\sigma_{Ya_6} + \sigma_{a_1a_4} + \\ &X\cdot\sigma_{a_1a_5} + Y\cdot\sigma_{a_1a_6} + X\cdot\sigma_{a_2a_4} + X^2\cdot\sigma_{a_2a_5} + X\cdot Y\cdot\sigma_{a_2a_6} + \\ &Y\cdot\sigma_{a_3a_4} + X\cdot Y\cdot\sigma_{a_3a_5} + Y^2\cdot\sigma_{a_3a_6}\end{aligned} \quad (10)$$

where

a_i: transformation coefficients $i = 1, 2, ..., 6$.

X, Y: coordinates in the previous reference system

The horizontal position errors with the σ_o (a–posteriori) of the transformation and the table of variance–covariance of the adjustment may be the quality index of the ACT.

Following the examination in variance–covariance V_x tables of the model's adjustment results, it was observed that the σ_{aiaj} (covariances), are in fact zeros (-0-). More precisely, they present values from 10^{-7} to 10^{-8} or even smaller, which may be considered null. However, the terms are preserved and used in the equations in order to avoid simplifications, which may not be valid in certain cases.

Given the perpendicularity principle of the components of the error vector, the correlation between σ_X and σ_Y is a unit (-1-) because the orientation of σ_r is given (vertical to the contour) (Achilleos, 2002; Keefer et al.,1988).

Therefore:

$$\rho_{XY} = 1 = \frac{\sigma_{XY}}{\sigma_X\cdot\sigma_Y} \Rightarrow \sigma_{XY} = \sigma_X\cdot\sigma_Y$$

The size of the covariance σ_{XY} (prior to the transformation) and its participation in the equations 9 (9) are negligible, because this size is multiplied by a quantity of 10^{-6} and smaller; these terms are also preserved in the equations. The terms σ_{Xa_i}, σ_{Ya_i} do not exist, because the coordinates X, Y of the points in the previous reference system for transformation did not take part in the adjustment of a_i in the transformation model. Finally, equations 9 (9), under these assumptions and conditions, may be simplified in the type (11):

$$\sigma_x^2 = a_2^2 \cdot \sigma_X^2 + a_3^2 \cdot \sigma_Y^2 + \sigma_{a_1}^2 + X^2 \cdot \sigma_{a_2}^2 + Y^2 \cdot \sigma_{a_3}^2 + 2 \cdot X \cdot \sigma_{a_1 a_2} + 2 \cdot Y \cdot \sigma_{a_1 a_3} + 2 \cdot X \cdot Y \cdot \sigma_{a_2 a_3}$$

$$\sigma_y^2 = a_5^2 \cdot \sigma_X^2 + a_6^2 \cdot \sigma_Y^2 + \sigma_{a_4}^2 + X^2 \cdot \sigma_{a_5}^2 + Y^2 \cdot \sigma_{a_6}^2 + 2 \cdot X \cdot \sigma_{a_4 a_5} + 2 \cdot Y \cdot \sigma_{a_4 a_6} + 2 \cdot X \cdot Y \cdot \sigma_{a_5 a_6}$$

$$\sigma_{xy} = a_2 \cdot a_5 \cdot \sigma_X^2 + a_3 \cdot a_6 \cdot \sigma_Y^2 + (a_2 \cdot a_6 + a_3 \cdot a_5) \cdot \sigma_{XY} + \sigma_{a_1 a_4} + X \cdot \sigma_{a_1 a_5} + Y \cdot \sigma_{a_1 a_6} + X \cdot \sigma_{a_2 a_4} + X^2 \cdot \sigma_{a_2 a_5} + X \cdot Y \cdot \sigma_{a_2 a_6} + Y \cdot \sigma_{a_3 a_4} + X \cdot Y \cdot \sigma_{a_3 a_5} + Y^2 \cdot \sigma_{a_3 a_6} \quad (11)$$

Usually we cannot know the initial errors in the *X* and *Y* axes that were made in field or office measurements, on map sheets design, the errors in printing, digitization and adaptation to the HATT projection system (Thapa et al., 1992). Given this problem, it is usually acknowledged that the initial error of the process is constant and related to the distinctive quality of the map scale that is used, to the map digitization procedure and its adaptation to the reference system of the HATT map projection.

If we take into account that σ_X and σ_Y are constant, the following terms of equations 11 (11) may be considered constant, i.e. that they do not depend on the point's position (12):

$$const(x) = a_2^2 \cdot \sigma_X^2 + a_3^2 \cdot \sigma_Y^2 + \sigma_{a_1}^2 + 2 \cdot a_2 \cdot a_3 \cdot \sigma_{XY}$$
$$const(y) = a_5^2 \cdot \sigma_X^2 + a_6^2 \cdot \sigma_Y^2 + \sigma_{a_4}^2 + 2 \cdot a_5 \cdot a_6 \cdot \sigma_{XY} \quad (12)$$

It would be interesting to note that the bigger the initial errors σ_X and σ_Y are, the more the resultant horizontal error after the transformation application (σ_r) tends to be stabilised at a high value, but with small relative deviations and range. Table 2 presents the results of the related investigation on this matter.

Table 2. Value range σ_r for various values of σ_X and σ_Y

σ_X, σ_Y	Value range σ_r	Difference
1	15.0–19.0	4.0
10	26.0–29.0	3.0
20	47.0–48.5	1.5
50	112.0–112.5	0.5

This indicates that the effect of the transformation on data quality is relatively much higher on data that initially presents very good accuracy, than on data presenting a low initial accuracy. Therefore, attention should be drawn when using the model in cases where data is of good accuracy and quality; these properties must be preserved (Figure 4).

The resultant horizontal position error σ_r, which is calculated by equations 14 (14), is the total error, which was propagated to the data up to that point of the process.

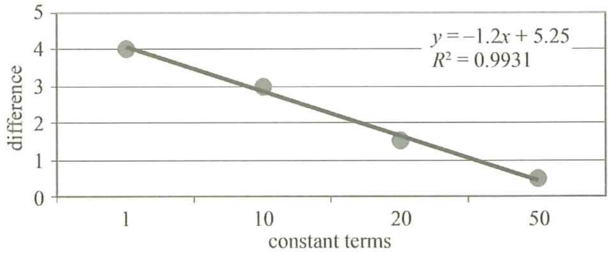

Figure 4. Investigation of the effect of the constant terms (12) on equations (11)

IV. INVESTIGATION DESIGN

In the following investigation we use a map sheet of the island of Crete (VATOLAKKOS) in a 1:50.000 scale and at HATT projection (Figure 5). On this map sheet the 20m contours were digitized and the Affine Transformation was applied in order to transform the digital contours from the HATT system to the Greek Geodetic Reference System 1987 (EGSA'87), where the errors caused by the transformation are investigated. For this reason, we used trigonometric control points from the national trigonometric network with their respective coordinates in the HATT and EGSA'87 projection systems.

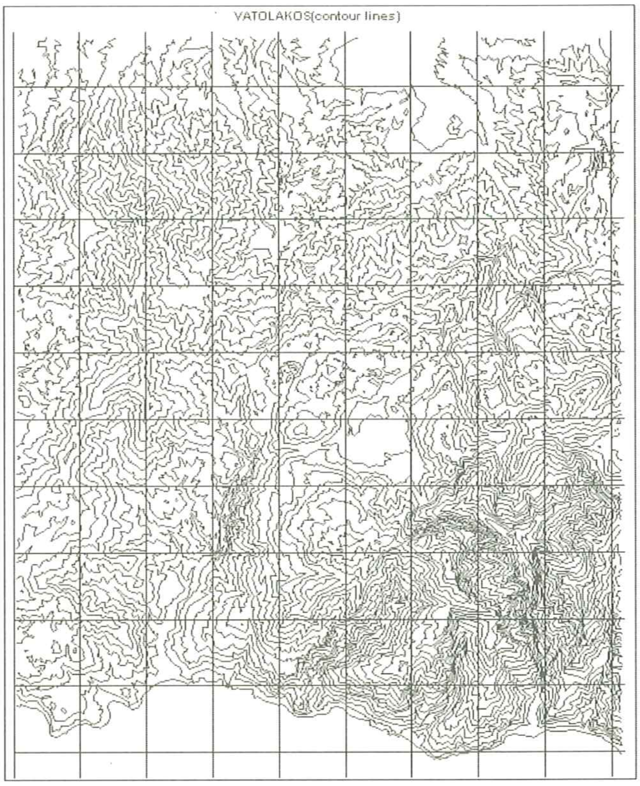

Figure 5. 1:50.000 map sheet VATOLAKKOS (source: Hellenic Military Geographical Service, HMGS)

As it is already mentioned, because it is not possible to know the initial errors in the *X* and *Y* axes that were made in field or office measurements, on the map sheets design, the errors in printing, digitization and adaptation to the HATT projection system (Thapa et al., 1992), it is accepted that the initial error of the analogical map is constant and equal to the distinctive quality in the given scale, i.e. $\sigma_K = 12.5$m. This error may be analysed in two components in the X (σ_{KX}) and Y axes (σ_{KY}), also by accepting that these two components are equal. Therefore:

$$\sigma_{KX} = \sigma_{KY} = \sqrt{\frac{\sigma_K^2}{2}} = \pm 8.84\text{m}$$

Following this, the contours digitization error depends on: a) the error made by the user during the digitization (σ_d); and b) the error made due to the digitizer's resolution (σ_μ) (Nakos, 1990). These errors are defined in this particular investigation to be $\sigma_d = \pm 12.5$m and $\sigma_\mu = \pm 5.0$m. That is, the final error (σ_ψ) only due to digitization, is:

$$\sigma_\psi = \sqrt{\sigma_d^2 + \sigma_\mu^2} = \pm 13.46\text{m}$$

Therefore, by the error propagation law it is derived that the final errors σ_X and σ_Y, which are considered to be initial for the investigation of the ATC, are:

$$\sigma_X = \sqrt{\sigma_{KX}^2 + \sigma_\Psi^2} = \pm 16.10\text{m}$$
$$\sigma_Y = \sqrt{\sigma_{K\Psi}^2 + \sigma_\Psi^2} = \pm 16.10\text{m} \quad (13)$$

The resultant horizontal position error σ_r, may be calculated by relations (14):

$$\sigma_r = \max(\sigma_X, \sigma_Y) \text{ or } \sigma_r = \text{SQR}(\sigma_X^2, \sigma_Y^2) \quad (14)$$

We preferred to use the second relation, because it gives a mean horizontal error cycle. The first relation overestimates the error at each point, by generalizing it to the maximum. Therefore, from relations (13), (14) it is assumed that:

$$\sigma_r = \pm 22.77\text{m}$$

Moreover, one can use (σ_X, σ_Y) directly without the need to calculate the σ_r. Because errors (σ_X, σ_Y) are by acceptance equal between them, they can be generalized to σ_r. This is, however, something that is not in fact valid. This acceptance is important because it gives a value of a constant horizontal position error prior to the transformation, which provides the opportunity to understand the effect of the application of the transformation model on the data accuracy.

The process followed for investigating the effect of the ATC on data accuracy is:

- Based on the given coordinates of the trigonometric control points in both reference systems, the system of equations of ATC 1 (1) is solved by adjustment to determine the six coefficients α_i (i=1,2,···,6), their variances-covariances σ_{ai} (i=1,2,···,6), $\sigma_{ai\,aj}$(i=1,2,···,6 / j=i+1,···,6) and a-posteriori σ_o.
- Any other point with given coordinates (X,Y) in the HATT projection system is transformed with the equations system of ATC 1 (1) in order to calculate its position on the new EGSA'87 projection system (x,y), considering as initial errors the (σ_X, σ_Y) of relations 13 (13).
- By the error propagation law we estimated the components of the horizontal position error in the X and Y axes, after the ATC (σ_X, σ_Y).
- Using equation 14 (14) we estimated the resultant horizontal position error σ_r after the ATC.

With regard to the transformation control points, we investigated three cases, which are presented in detail:

INVESTIGATIONS:

1st Investigation
- Use of the four corners on the map sheet as transformation control points (usual case)

2nd Investigation
- Use of trigonometric control points with given coordinates (X, Y) and (x, y).

a) Using a limited number of trigonometric control points (4–6)
b) Using a big number of trigonometric control points (14–20)

3rd Investigation
- Use of trigonometric control points with given coordinates (X, Y) and (x, y) distributed in individual sections of the area transformed (partial solution)

A. 1st Investigation: use of the four corners on the map sheet

The four corners of the given map sheet are used as the transformation control points. This is a usual case in routine cartographic tasks.

The results are presented on Tables 3 and Table 4. Table 4 indicates that although very high maximum values of the horizontal position error are presented, its mean value is mean σ_r=±28.52m. This mean value is higher than initial error σ_r, which was considered to exist prior to the ATC (σ_r=±22.77m); however, this can be verified also by statistical control.

For 95% confidence level, the mean value of σ_r is within range (Hammond R., et al., 1974):

$$X_{\text{mean}} - Z_{0.95\%} \cdot \sigma X_{\text{mean}} \leqslant X_{\text{mean}} \leqslant X_{\text{mean}} + Z_{0.95\%} \cdot \sigma X_{\text{mean}}$$
$$Z_{0.95\%} = 1.96$$
$$X_{\text{mean}} = \text{mean } \sigma_r = 28.52 \text{ m}$$
$$\sigma X_{\text{mean}} = \text{sd } \sigma_r/\text{sqr}(N) = 0.1070 \text{ m}$$
$$\rightarrow 28.41 \leqslant \text{mean } \sigma_H \leqslant 28.63$$

Therefore, for 95% confidence level, the mean value σ_r after the ATC application is higher than its initial value (±22.77m).

A horizontal position error smaller than 25.00m, which could be observed in this scale is not present in the areas transformed.

The following map (Figure 6) shows, with regard to the map sheet VATOLAKKOS, the map of Horizontal Position Error σ_r. The error's values are in the form of equivalent curves (equi-error curves), while areas with a big error appear in bright colours.

In brief, the use of the four corners of the map sheet in the ATC does not seem to produce satisfactory results, in relation to the horizontal position error. As it will be shown further

Table 3. Transformation Table for the Investigations 1, 2a, 2b

	Case 1 (Four corners)	Case 2a (Few points)	Case 2b (Multiple points)
a_1	0.0047±9.11947	−0.0162±4.66796	0.0095±2.13483
a_2	0.9998±0.0008	0.9998±0.00073	1.0013±0.00032
a_3	0.0004±0.00066	0.0007±0.00054	−0.0001±0.00026
a_4	0.2499±9.11947	0.0002±4.66796	0.6521±2.13483
a_5	0.0010±0.00080	−0.0019±0.00073	−0.0001±0.00032
a_6	1.0000±0.00066	1.0002±0.00054	1.0013±0.00026
x_1	−8505.000	−6625.145	−8860.132
y_1	9927.500	5745.982	9689.011
x_2	488441.09	490346.15	488096.21
y_2	3910533.75	3906321.75	3910275.00
σ_{apost}	±18.23895	±11.43411	±9.05730
rotation	−0.01952 grad	0.06681 grad	0.03301 grad
scale	1.00475	1.00408	1.00285
RMS	25.78640	27.91925	49.50208

Table 4. Table of results for the Investigations 1, 2a, 2b

"VATOLAKKOS"			
	Case 1	Case 2a	Case 2b
min σ_r(m)	±26.19	±23.70	±23.00
max σ_r(m)	±31.94	±30.46	±24.02
mean σ_r(m)	±28.52	±25.34	±23.32
sd σ_r(m)	±1.42	±1.35	±0.21
50% of area	≤±28.39m	≤±25.01m	≤±23.30m
75% of area	≤±29.51m	≤±26.01m	≤±23.46m
90% of area	≤±30.30m	≤±27.08m	≤±23.61m
σ_r≤±25 m	0.00%	48.48%	100.00%
σ_r≤±30m	84.09%	99.39%	100.00%
σ_r≤±35m	100.00%	100.00%	100.00%

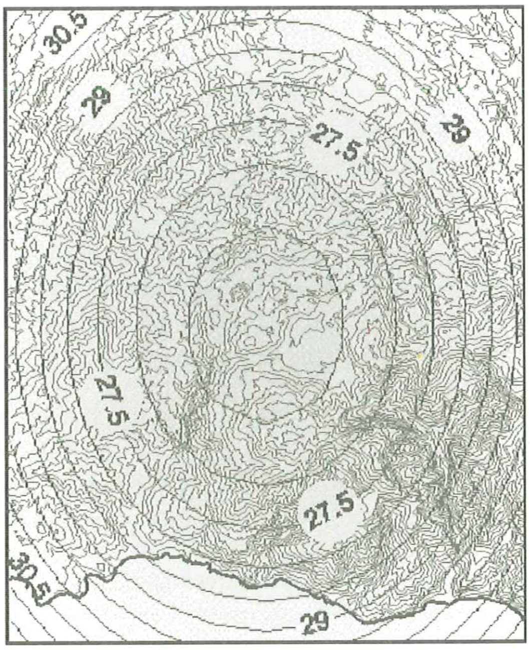

Figure 6. Map of horizontal position error σ_r for the Investigation 1

down, the results are improved significantly when using the trigonometric control points as the transformation control points.

B. 2nd Investigation: use of trigonometric control points

In this investigation we examine two cases: a) the use of a limited number of trigonometric control points as transformation control points; and b) the use of a big number of trigonometric control points. Besides, it is common to have a "few" trigonometric points in an area in question which is to be transformed.

The results of the transformation and the analysis of the horizontal errors of the map sheet of this case (map sheet VATOLAKKOS) are presented on Tables 3, Table 4.

It is noted that in case 2a six (-6-) trigonometric control points were used and in case 2b twenty (-20-) points. In both cases an effort was made to have a uniform distribution of the trigonometric control points in the area transformed.

The results of these two cases appear schematically on the maps of Figure 7 and Figure 8.

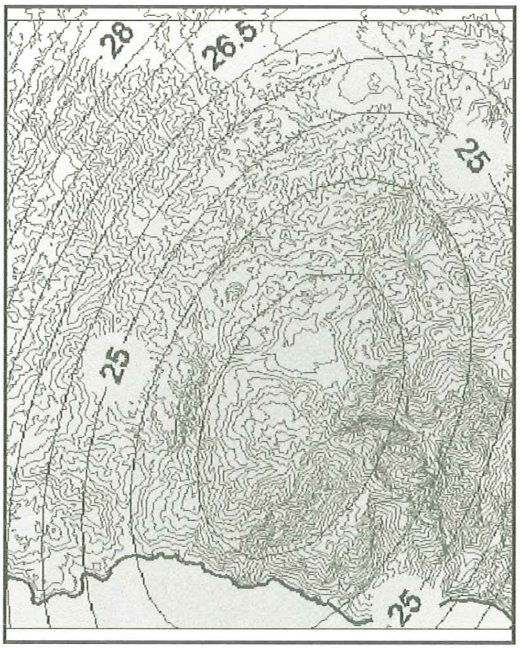

Figure 7. Map of horizontal position error σ_r for Investigation 2a

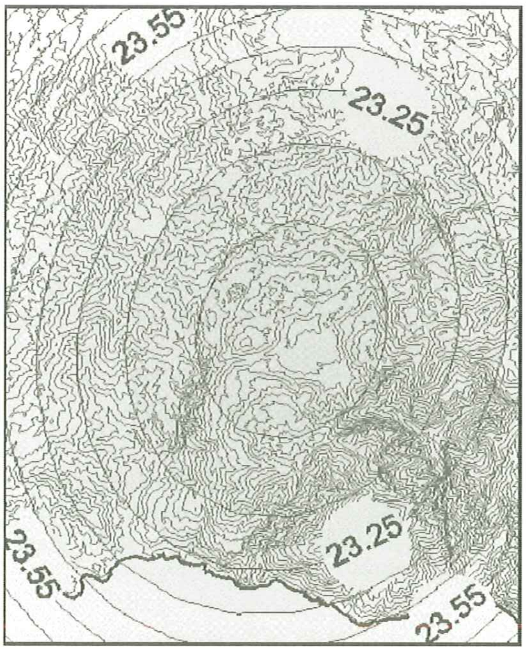

Figure 8. Map of horizontal position error σ_r for Investigation 2b

Based on the results, we observe that in the case of the limited number of trigonometric control points, the maximum value of σ_r(max σ_r=±30.46) is higher compared to the case of the multiple trigonometric control points (max σ_r=±24.02). However, these values are high in both cases with regard to the accuracy of the scale 1:50.000 (in particular the value in investigation 2a).

If these values are compared to the horizontal position error prior to the application of the ATC (σ_r=±22.77m), it is observed that the value of σ_r in the first case is relatively high, while in the second case it is good enough.

When observing the mean horizontal errors (mean σ_r) and their standard deviations(sd σ_r) it appears that these results are much better in comparison to the first investigation. In the first case, the mean σ_r is ±25.34, and in the second case ±23.32, which is much better.

The area percentage that presents horizontal position errors smaller than 25.00m, is 48.48% in the first case and 100.00% in the second case 2b.

Moreover, 90% of the transformed area has a horizontal position error smaller than ±27.08m, and in the second case smaller than ±23.61m.

There is a clear improvement of the results in the conditions of the second investigation (α, β) in comparison to the first investigation. The possibility to use even a "few" trigonometric control points improves very much the quality of the results from the ATC application.

C. 3rd Investigation: partial solution

The ATC is the attempt to adapt a linear model to a group of data. If the extent of the area examined is large, then this adaptation may not be satisfactory. In these cases it is usually required either to change the model, or to apply partially the linear model in sub-areas on the map.

In this investigation, the map sheet (VATOLAKKOS) is divided into four equal sections, with the number of the trigonometric control points, while an effort was made to use equal number of trigonometric control points in each section. The parameters of the transformation as well as the brief results of the horizontal error σ_r of each section are presented on Table 5, Table 6.

The maps (Figure 9) present horizontal position errors for the four individual areas of the of the study area. The horizontal error σ_r is depicted in the form of equivalent curves (equi-error curves), and the areas with small horizontal position error appear in bright colours.

Figure 9 shows the elliptical form of the horizontal position error σ_r. The error appears to increase in the areas, which are the most distant from the center of the ellipses. It is therefore

Table 5. Transformation table for investigations 3a, 3b, 3c, 3d

	Case 3a (Section 1)	Case 3b (Section 2)	Case 3c (Section 3)	Case 3d (Section 4)
a_1	0.0227±3.61948	−0.0182±2.27864	0.0076±5.90253	−0.0071±4.81799
a_2	1.0099±0.00094	0.9991±0.00080	1.0006±0.00182	1.0040±0.00112
a_3	-0.0021±0.00119	−0.0001±0.00074	−0.0025±0.00184	−0.0003±0.00141
a_4	0.0635±3.61948	0.0501±2.27864	−0.2498±5.90253	0.1001±4.81799
a_5	0.0026±0.00094	0.0017±0.00080	0.0025±0.00182	−0.0003±0.00112
a_6	0.9998±0.00119	1.0005±0.00074	0.9997±0.00184	1.0004±0.00141
x_1	−13858.364	−3788.349	−2971.006	−3689.42
y_1	18212.746	16011.052	2342.162	2425.463
x_2	483072.656	493184.06	494011.78	482294.81
y_2	3918830.500	3916629.25	3902894.00	3902983.50
σ_{apost}	±7.23897	±5.09520	±11.80507	±10.77334
rotation	−0.04376 grad	−0.03739 grad	−0.10937 grad	0.03591 grad
scale	1.00496	1.00411	1.00306	1.00163
RMS	10.31188	10.21292	16.71139	21.54988

Table 6. Table of results for investigations 3a, 3b, 3c, 3d, 3entire

"VATOLAKKOS"					
	Case 3a	Case 3b	Case 3c	Case 3d	Case 3(entire)
min σ_r(m)	±23.01	±23.35	±24.28	±23.81	±23.00
max σ_r(m)	±25.04	±25.97	±33.45	±26.36	±33.45
mean σ_r(m)	±23.48	±24.07	±26.71	±24.68	±24.76
sd σ_r(m)	±0.43	±0.51	±1.82	±0.56	±1.61
50% of area	⩽±23.35m	⩽±24.00m	⩽±26.39m	⩽±24.60m	⩽±23.51m
75% of area	⩽±23.67m	⩽±24.33m	⩽±27.49m	⩽±25.04m	⩽±25.22m
90% of area	⩽±24.11m	⩽±24.62m	⩽±28.98m	⩽±25.41m	⩽±26.88m
σ_r⩽±25 m	99.39%	94.31%	17.61%	70.00%	69.90%
σ_r⩽±30m	100.00%	100.00%	93.75%	100.00%	98.35%
σ_r⩽±35m	100.00%	100.00%	100.00%	100.00%	100.00%

possible to predict in advance where big error will be present after the creation of a DTM (the areas on the map periphery may present the biggest errors).

The mean horizontal position error σr is ±25.91m for the entire map and the standard deviation of the mean error is ±2.08m. 90% of the area in question presents a horizontal position error σ_r smaller than ±28.72m, while 45.32% a horizontal position error smaller than ±25.00m.

The overall results of the horizontal error σ_r, on the entire map, are presented on Table 6 (Table 6).

In brief, this investigation gives good enough results in relation to Investigation 1 and 2a, but not better than the ones in Investigation 2b. This happens because the division of the area in question into sub-areas distributes the trigonometric control points in these sub-areas. Therefore, the trigonometric control points that appear in each sub-area are few (4–6) as in the case of Investigation 2a (6). Therefore, the partial adaptation of the linear model is much better than the adaptation on the entire map (Investigation 2a).

D. Comments on the three investigations

The constant terms that appear in the calculation equations of σ_X, σ_Y(12) participate at 50%–100%. The size of these terms for the three investigations is presented on the following Table of Constant Terms (Table 7).

The participation percentage of the constant terms is calculated on the basis of max σ_r, which appears on the Transformation

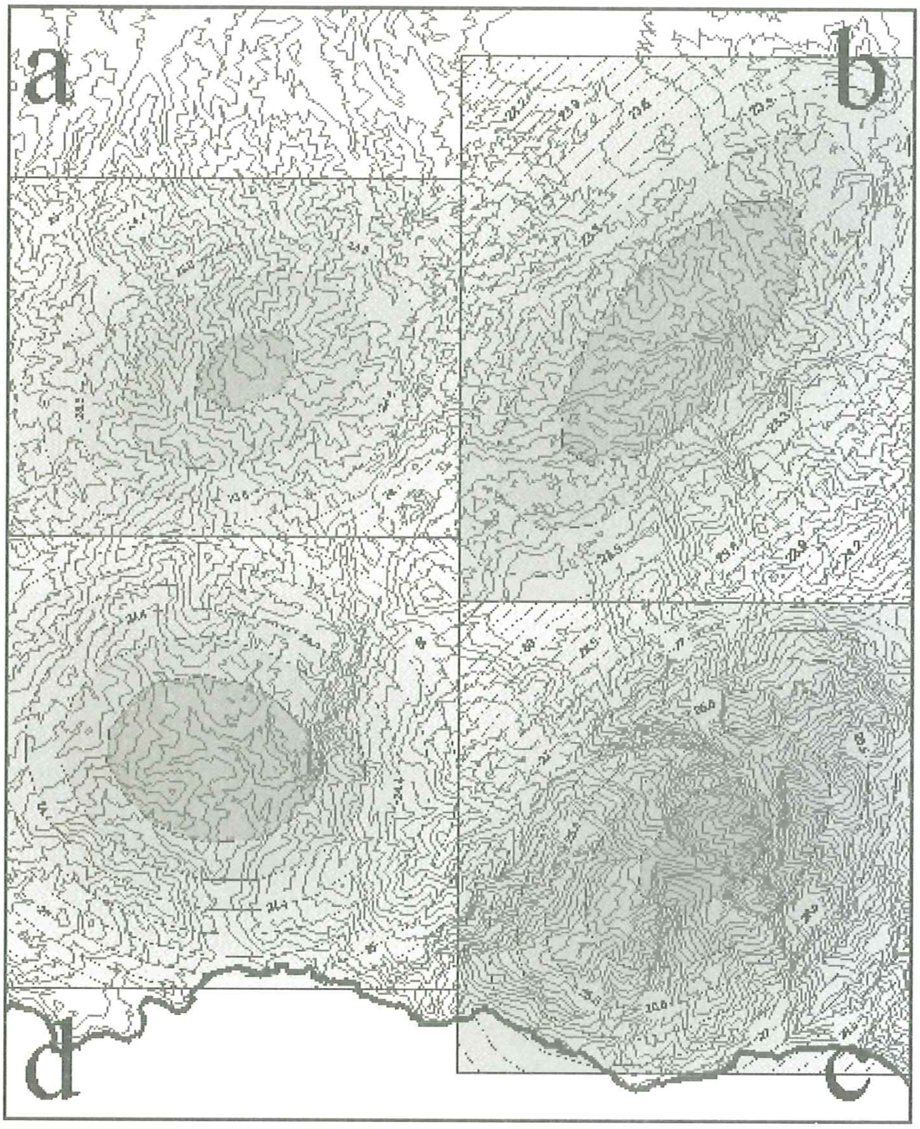

Figure 9. Map of horizontal position error σ_r for individual area of investigation 3 (sub-areas 3a, 3b, 3c, 3d)

Table 7. Table of constant terms

Investigation	const(X)	const(Y)	Participation percentage in max σ_r(%)	R.M.S.E
1	437.20	436.24	63	34.938
2a	305.26	305.54	63	27.919
2b	288.15	288.15	97	49.502
3a	297.73	297.05	70	10.312
3b	288.25	289.04	53	10.213
3c	318.35	317.84	59	16.711
3d	307.73	305.70	71	21.550

Tables of the previous investigations.

Given the horizontal position error prior to the ATC (σ_r= ±22.77m), the maximum permissible RMSE ($RMSE_f$) may be calculated; if surpassed, the transformation will not be accepted for confidence level 95%. $RMSE_f$ derives from relation (Jones J R., 1991):

$$RMSE_f = \sigma_r / Z_{0.95\%}$$

The $Z_{0.95\%}$ value is 1.96 therefore $RMSE_f$ is 11.62m. Based on this critical $RMSE_f$, only the transformations in investigations 3a and 3b are acceptable at 95% confidence level. Moreover, relatively good results appear in case 2a, where a few trigonometric control points were used.

It is however noted that in none of the three cases the US National Map Accuracy Standard (US Bureau of the Budget) is observed. According to this standard:

"90% of points tested shall be in error not more than 1/50 inch for maps on publication scales of 1:20,000 or smaller"(Fisher, 1991)

This standard on the ground means 25.4m and is observed only in case 2b (90% of horizontal errors σ_r are smaller than 23. 61m). The cases of Investigations 2a and 3 approach the standard, but do not observe it.

Therefore, errors imported in the data due to the application of the ATC cannot be ignored, but in certain cases are important; for example when data transformed presents high accuracy and when it is useful to know them for defining later on the result's accuracy.

It is generally believed that with regard to GTC there is no reason to worry as to accuracy matters in its results. This is because in case it is observed that the result of the coordinates transformation is not satisfactory in terms of accuracy, then there is the possibility of improvement. This does not mean that knowing the error propagation model, due to the application of the ATC (and GTC in general) is not of interest. There are data which is already transformed and for which we know their transformation parameters. It is useful to have a model, which provides an estimation of the quality and accuracy of the results of the transformation application.

V. EFFECT OF A GTC MODEL ON THE ELEVATION ACCURACY OF A DEM

The effect of a model of GTC on the accuracy of data's horizontal position is evident. If this data is used in further processing, this effect is transferred to the derivatives of this data.

Contours (primary information for creating a DEM) present horizontal position errors. These errors, together with errors resulting from the transformation model application determine the horizontal position error of contours.

The horizontal position error in contours is transferred in DEM during the interpolation process, in the form of elevation errors.

Koppe (Imhof, 1982) worked empirically in order to define the error due to the horizontal position of the contours in the derivative products. The relation he gave (15) connects the horizontal position error to the elevation error and depends on the relief's morphology in the area around the contour (ground slope):

$$\sigma_H = \sigma_r \cdot \tan(\alpha) \qquad (15)$$

where, α is the value of the ground slope at the point for which the elevation error is calculated and σ_r is the horizontal error.

This dependency is also depicted in Figure 10.

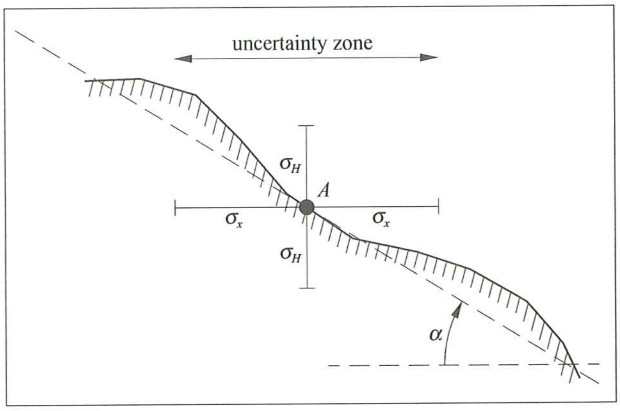

Figure 10. Elevation error due to horizontal position error

VI. INVESTIGATING CORRELATIONS OF THE RESULTS OF THE GTC TO THE ELEVATION ERRORS IN DEM

Given the results of the investigations of the GTC, we may attempt to investigate whether there are correlations of these results to elevation errors. Elevation errors are caused, using the transformed contours in an interpolation process. They result from the horizontal errors, which are produced when applying the GTC and to the interpolation process used.

With regard to the correlation effort in question, the calculation of the elevation error is based on the relation of Imhof (15) (Imhof, 1982):

The results from the application of the GTC and the interpolation on the data, for seven (7) cases investigated, are presented on Table 8.

This Table shows that case 1 presents the biggest horizontal and elevation errors. This makes sense, since this case applies the GTC only on four points, which are not trigonometric control points with given coordinates, but the four corners of the map sheet which is transformed.

Further on, the correlation indices are calculated among the parameters of Table 8, and are presented on Table 9 .

Table 9 shows a high positive correlation among the parameters:

max σ_r	max σ_H	0.94
σ_0	max σ_H	0.75
mean σ_0	max σ_r	0.91
σ_0	max σ_r	0.84
sd σ_0	max σ_r	0.93

Table 8. Overall results of the GTC application and interpolation

Case	1	2	3	4	5	6	7
σ_0	24.691	11.458	9.054	7.258	5.104	11.838	10.802
RMS	34.919	28.067	49.589	10.264	10.209	16.741	21.605
maxσ_r	37.220	31.070	23.960	29.110	26.410	32.840	29.370
minσ_H	0.000	0.000	0.000	0.000	0.000	0.000	0.000
maxσ_H	34.730	28.410	14.280	22.140	23.090	26.410	24.180
meanσ_H	9.680	8.410	6.900	6.880	6.610	8.370	7.020
sd σ_H	7.660	6.480	4.610	4.920	5.040	6.350	5.120
cp	4	6	20	5	5	6	4

Table 9. Parameters correlation indices of Table 8

	maxσ_H	maxσ_r	meanσ_H	minσ_H	RMS	σ_0	sd σ_H	cp
maxσ_H	1.0000							
maxσ_r	0.9403	1.0000						
meanσ_H	0.8427	0.9071	1.0000					
minσ_H	–	–	–	1.00				
RMS	–0.1787	–0.0776	0.2766	–	1.0000			
σ_0	0.7573	0.8436	0.9048	–	0.4191	1.0000		
sd σ_H	0.9177	0.9310	0.9825	–	0.1365	0.8690	1.0000	
cp	–0.7487	–0.6282	-0.3019	–	0.7426	–0.2222	–0.4419	1.0000

σ_0	meanσ_H	0.84
sd σ_H	meanσ_H	0.93
cp	RMS	0.74
sd σ_H	σ_0	0.87

A high negative correlation is observed between the parameters:

cp	maxσ_H	–0.75
cp	maxσ_r	–0.63

while even smaller correlations are observed among the parameters, which are not strong, but indicate a clear tendency (Table 9).

The following diagrams present these correlations, which exist and apply in the parameters of Table 8 (Figure 11–Figure 17).

An important remark can be concluded from Table 8 and Figure 13–Figure 17 when examining case 3 in comparison to cases 4 and 5; it is observed that both groups of cases present similar results of mean elevation error and standard deviation of elevation error, while they differ greatly with regard to the random root-mean-square error (RMS) of the GTC (R.M.S.) and the maximum horizontal and elevation error.

From the combination of this data we may conclude that the use of many control points during the application of GTC and, mainly, the use of trigonometric control points to this regard, improves the GTC model's adaptation to the application data

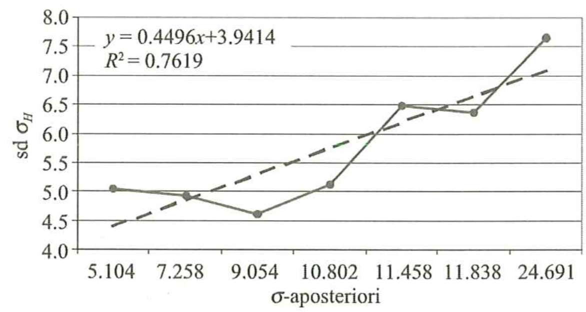

Figure 11. Linear correlation between the aposteriori error of the GTC application and the mean elevation error

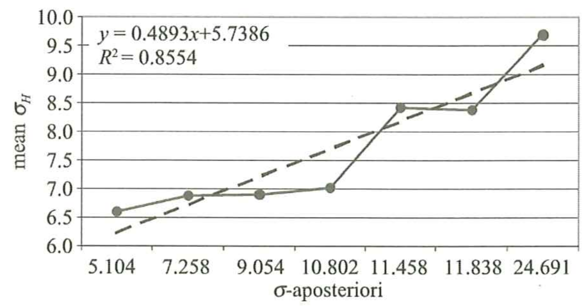

Figure 12. Linear correlation of aposteriori error of the GTC application and the standard deviation of elevation error

(max σ_r), but does not produce any remarkable improvement in essential results, in our case in elevation errors.

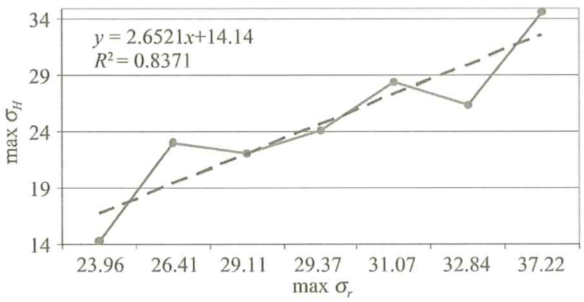

Figure 13. Linear correlation of maximum horizontal error of the GTC application and the maximum elevation error

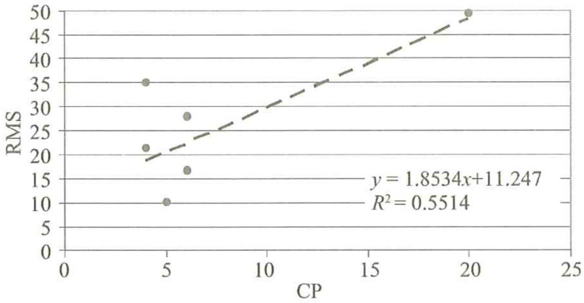

Figure 14. Linear correlation between the number of control points of the GTC application and the random R.M.S. of the GTC

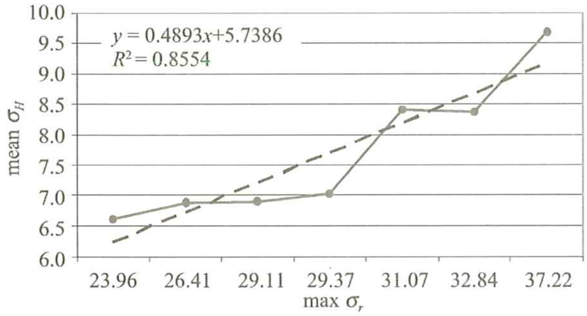

Figure 15. Linear correlation between the maximum horizontal error of the GTC application and the mean elevation error

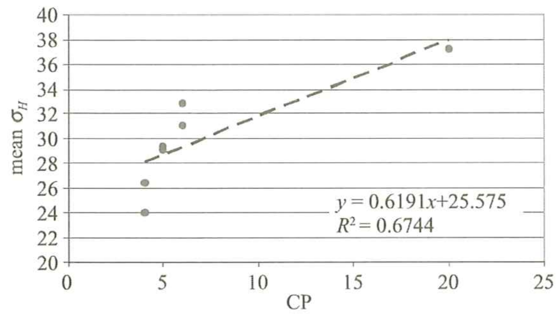

Figure 16. Linear correlation between the control points of the GTC application and the maximum horizontal error

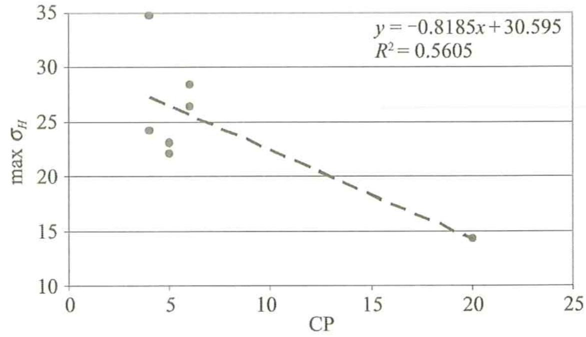

Figure 17. Linear correlation between the number of control points of the GTC application and the maximum elevation error

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