PDEMR Modelling of Protea Species in the Population Size of 1 to 10, in Cape Floristic Region from 1992 to 2002, South Africa

Danni Guo[1], Renkuan Guo[2], Guy F. Midgley[1], Rebelo A. G. [1]

[1]Kirstenbosch Research Center, South African National Biodiversity Institute, Private Bag X7, Claremont 7735, Cape Town, South Africa
[2]Department of Statistical Sciences, University of Cape Town, Private Bag, Rondebosch 7701, Cape Town, South Africa

Abstract
Global warming and climate changes can lead to the movement of plant species as they find their original habitats are no longer suitable to their needs. It is often an urgent task to establish a mathematical model to catch up the trajectories of the endangered species to effectively manage environmental protection under the inevitable biodiversity changes taking place. However, as it often happens with the environmental data, within the study area, some areas are well sampled, while other areas are not sampled. Even the collected data are often just species presence or categorical data. This makes very difficult to a spatial analysis, and impossible to do a kriging prediction map. In this paper, we use the *partial differential equation motivated regression* (PDEMR) model, to model Protea species in the population size of 1 to 10, in the Cape Floristic Region, from 1992 to 2002, in South Africa.
Keywords
Partial Differential Equation Motivated Regression (PDEMR), mathematical model, Protea, endangered species, global warming, climate changes, Cape Floristic Region

I. INTRODUCTION

Global warming and climate changes are changing the environment and therefore changing the distribution and behaviour of the plant species. Plant species often move and change their distributions as they find their original habitats are no longer suitable to their needs. The question that concerns us is: are the plant species distributions changing and how? It is therefore important to establish a mathematical model to catch up the movement and patterns of the endangered species in order to effectively manage environmental protection under the inevitable biodiversity changes that are taking place.

In reality, when plant samples are collected, it is often done with other important environmental and scientific purposes in mind, and was not intended for spatial predictions in the first place. Since the plant samples are not designed for spatial predictions, the samples are not well spread over the study area, and can not be used for spatial predictions, such as kriging. Another problem is that quite often the data collected are just species presence data or categorical data, and this makes very difficult to model the plants, and impossible to do a kriging prediction map.

In this paper, we will model the Protea species in the population size of 1 to 10, in the Cape Floristic Region, from 1992 to 2002, in South Africa. We are faced with two problems here: presence data only, and incomplete sample data. To solve first problem, we will use a simple technique in order to look at occurrence counts or frequency distributions of the Protea. To solve second problem, we will use the *partial differential equation motivated regression* (PDEMR) model, to fill in missing samples within the Cape Floristic Region. This paper will show the detailed steps and workings of the PDEMR model and how it helps in modelling plant distributions.

II. PROTEAS IN THE CAPE FLORISTIC REGION

The Cape Floristic Region is located at the southern tip of the Africa, and it covers parts of Western and Eastern Cape provinces of South Africa. It is home to some 9030 plant species, and nearly 70% of which are found nowhere else. Fynbos is the predominate ecosystem in the Cape Floristic Region, and it is under serious threat (Freeth et al., 2007).

The Protea Atlas Project collected samples of Fynbos's flowering Proteas in the Cape Floristic Region, South Africa, from 1992 to 2002. These sample data provides valuable information on the distribution and change in the Proteas. In this case, we are focusing on the category of Proteas, which have the estimated population size from 1 to 10, per sample site.

Figure 1 below shows the location of the Cape Floristic Region within South Africa, and Figure 2 shows the locations of Proteas occurrence of the population size of 1 to 10, in the Cape Floristic Region, from 1992 to 2002.

As one can see from Figure 2, the sample locations are not well spread, since its original purpose was spatial predictions, but for scientific and biodiversity knowledge. The samples tended to focus in certain areas, while other areas are entirely un-sampled. This creates a problem for kriging predictions.

1082-4006/06/1202-67$5.00

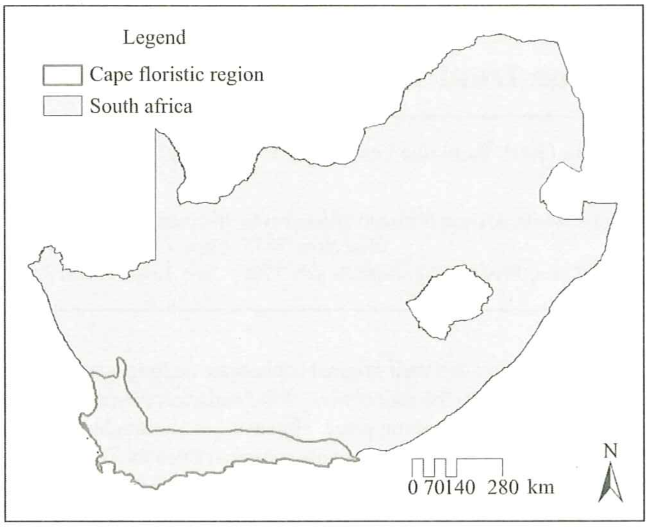


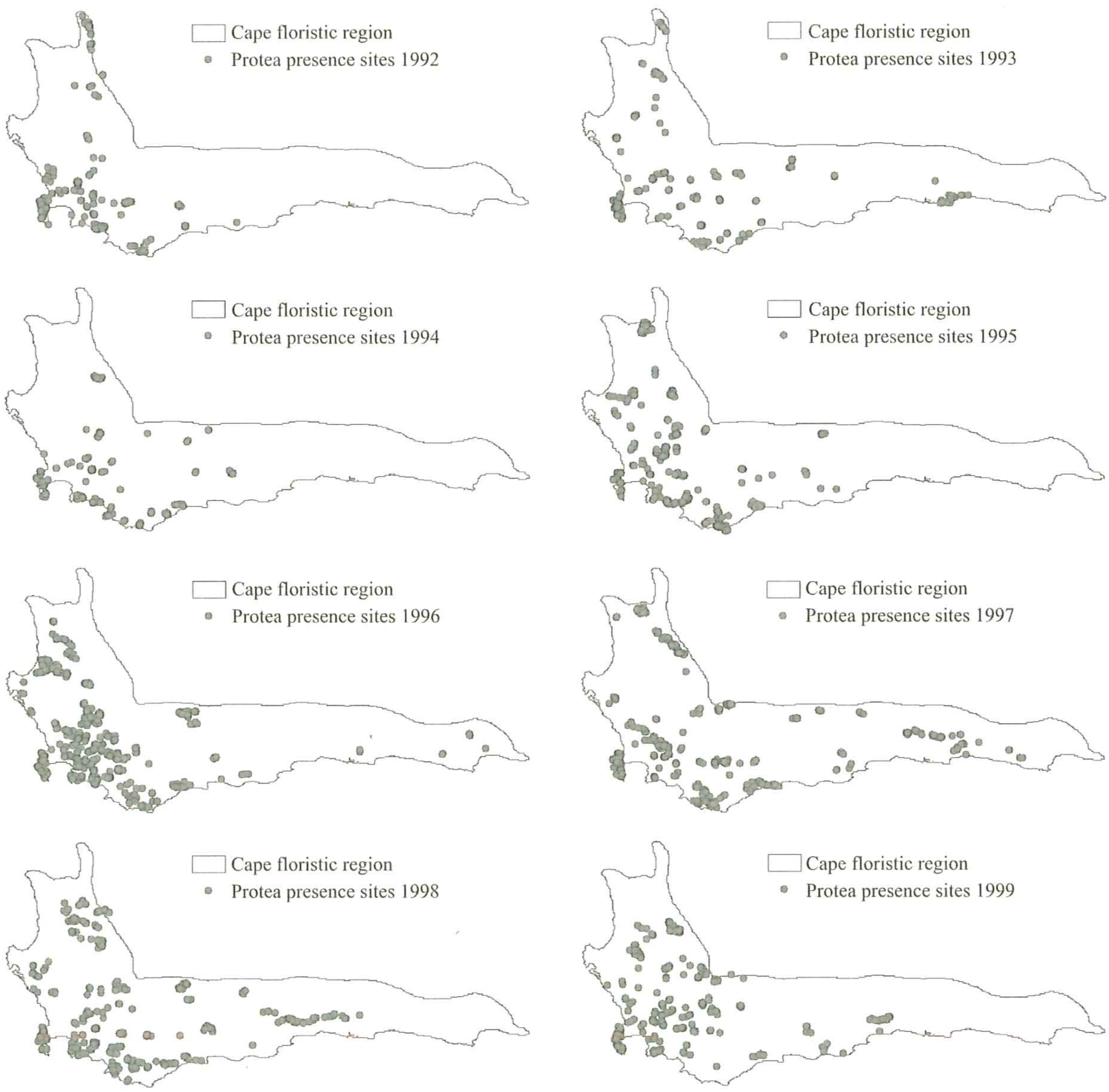
Figure 1. The cape floristic region within south africa

The Protea data are presence only data, and not numerical, which creates another problem for spatial analysis.

III. FREQUENCY COUNTS OF THE OCCURRENCE OF PROTEAS

To solve the problem of presence data only, this being a categorical data issue, a simple technique of using frequency counts is used. The Cape Floristic Region is divided into 243 grid cells, and within each cell, the presence of Protea species is counted, and the resulting value is attached to each centroid point of each cell. The centroid point is needed in order for kriging prediction maps to be produced. See Figure 3.

In Figure 4, the pink color are 0 in value, it shows the cells that does not have any frequency counts at all. In other words, the pink point cells show the un-sampled locations within the Cape Floristic Region. It is clear that a lot of the areas are



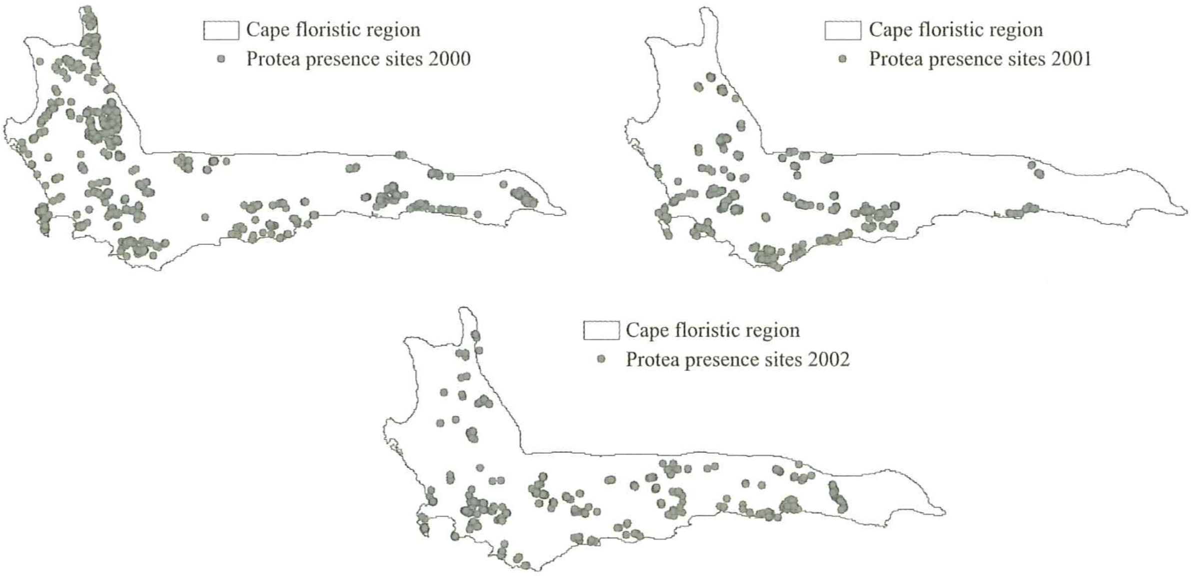


Figure 2. The sample locations of proteas in the population size of 1–10, in the cape floristic region, 1992–2002

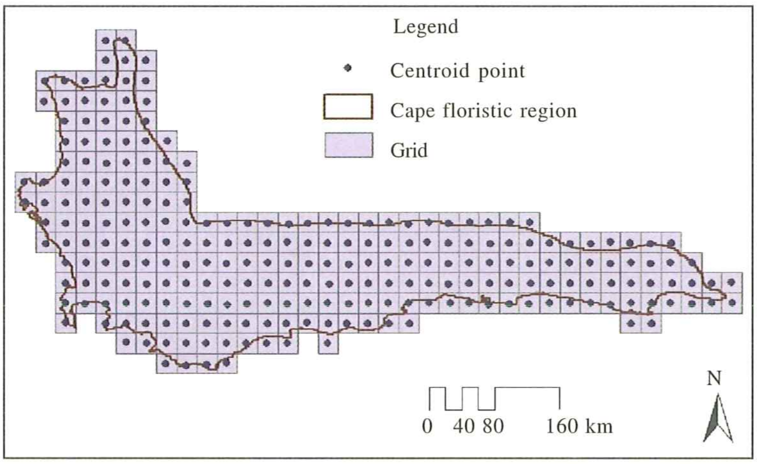


Figure 3. The grid cell division of cape floristic region

un-sampled, and these locations vary from year to year. In order for an accurate kriging prediction map to be produced, the missing cells must be filled. This means that the PDEMR model must be used in order to predict the un-sampled cells.

IV. THE CONCEPT OF DEMR MODEL AND THE COUPLING PRINCIPLE

In engineering theory, particularly, in modern control theory, it is often convenient to utilize a differential equation to describe the dynamic law of a continuous system. However, the unknown parameter vector θ associated with system

Definition 1: A pair of equations:

$$\begin{cases} \dfrac{d^{(p)}x}{dt^p} = \varphi\left(\dfrac{d^{(p-1)}x}{dt^{p-1}}, \dfrac{d^{(p-2)}x}{dt^{p-2}}, \cdots, x; \theta\right) & \text{(a)} \\ \dfrac{1}{h^p}\Delta x^{(p)}(k) = \varphi\left(\dfrac{1}{h^{p-1}}\Delta x^{(p-1)}(k), \dfrac{1}{h^{p-2}}\Delta x^{(p-2)}(k), \cdots, \hat{x}(k); \theta\right) + \varepsilon_k \quad k = 2, 3, \cdots, n & \text{(b)} \end{cases} \quad (1)$$

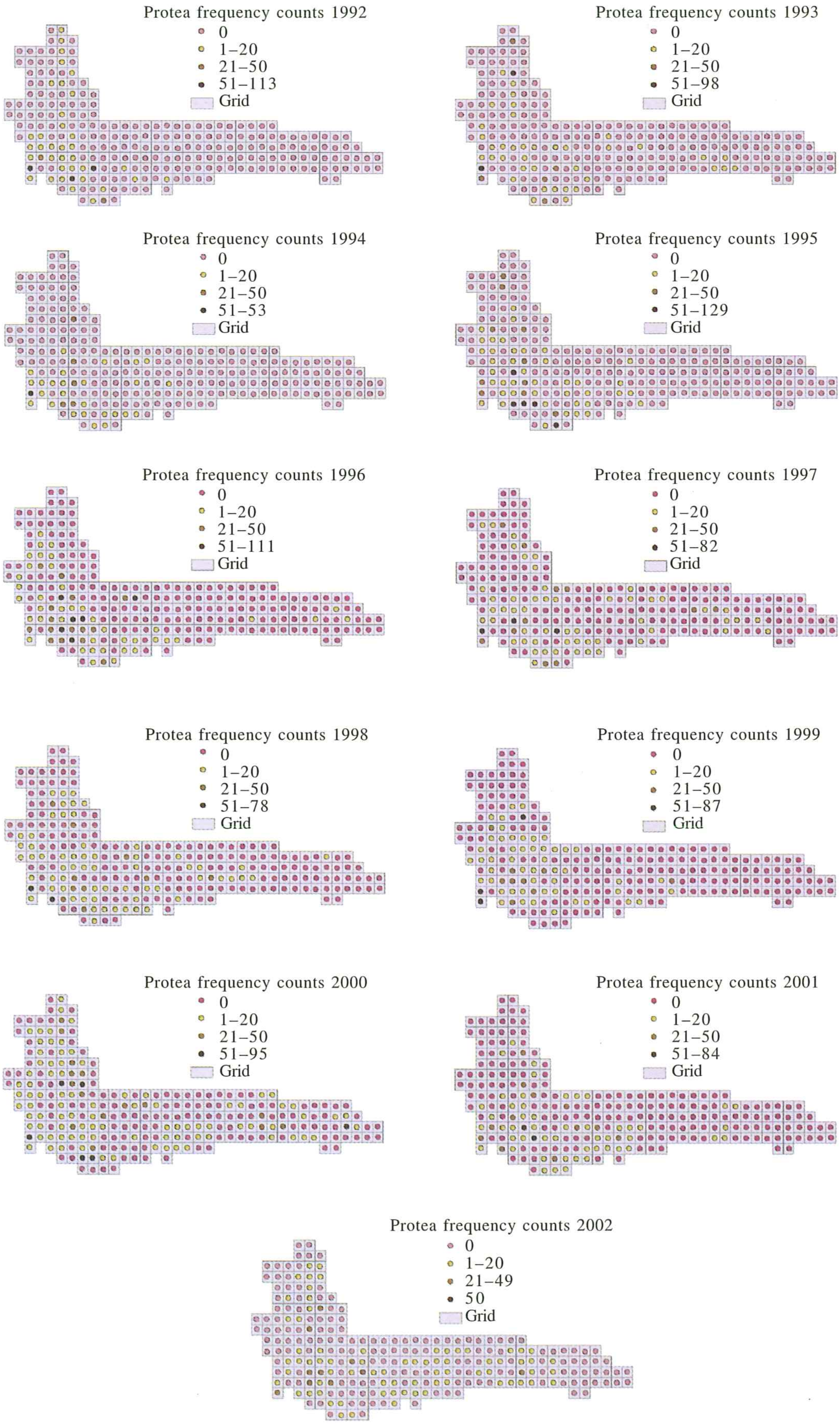


Figure 4. The Sampled Frequency Counts of Proteas in the Population Size of 1–10, in the Cape Floristic Region, 1992–2002

is called a p^{th}-order univariate differential equation motivated regression model, abbreviated as (p^{th}-order univariate) DEMR model. Eq. (1a) is called the motivated differential equation and Eq. (1b) is called the (first) coupled regression model, where h is the grid size for the first difference Δ. As to the term $\hat{x}(k)$ is the approximation to primitive function $x(t)$ at $t=k$.

If the observation on the system is at the first difference level, denoted as $\Delta X=\{\Delta x(1),\Delta x(2),\cdots,\Delta x(n)\}$ where $\Delta x(k)=x(k)-x(k-1)$. Note that the relation between summation operator $\sum$ and difference operator Δ, define $\Delta x(1) \triangleq x(1)$, then:

$$x(k)=\sum_{i=1}^{k}(\Delta x(i)) \qquad (2)$$

It is often using:

$$\hat{x}(k)=\frac{1}{2}[x(k)+x(k-1)] \qquad (3)$$

as a first approximation to $x(t)$ at $t=k$. Finally, $\{\varepsilon_k, k=2,3,\cdots,n\}$ is the error terms of the coupled regression model in Eq. (1b) paired in the above equation system Eq. (1). The nature of errors in Eq. (1) will be discussed later. For a better understanding, let us examine a simple example.

Example 2: Equation system:

$$\begin{cases} \dfrac{dx}{dt}=\alpha+\beta x & \text{(a)} \\ \Delta x(k)=\alpha+\beta\hat{x}(k)+\varepsilon_k \quad k=2,3,\cdots,n & \text{(b)} \end{cases} \qquad (4)$$

is the simplest first-order univariate DEMR model. Eq. (4b) is called as the coupled regression (abbreviated as CREG) model because its form strictly follows a "translation rule" based on the form of the motivated differential equation. We call this translation rule as the coupling principle in DEMR.

For an overall intuitive picture of DEMR model, we list the components and the translation rule in terms of the coupling principle in Table 1.

Table 1. Coupling rule in univariate first-order DEMR model

Term	Motivated DE	Coupled REG
	Translation rule between MDE and CREG	
Intrinsic feature	Continuous	Discrete
Independent variable	t	k
1st-order derivative	$dx(t)/dt$	$\Delta x(k) = x(k)-x(k-1)$
pst-order derivative	$d^{(p)}x(t)/dt^{p}$	$\Delta^{n}x(k) = \Delta^{n-1}x(k)-\Delta^{n-1}x(k-1)$
Primitive function	$x(t)$	$x(k)$
Model formation	$\dfrac{dx(t)}{dt}=\alpha+\beta x(t)$	$\Delta x(k)=\alpha+\beta\hat{x}(k)+\varepsilon_k$
	Parameter coupling	
Parameter	$(\mapsto, \updownarrow)$	(a,b)
Dynamics(Solution)	$x(t)=\left[x(0)-\dfrac{\alpha}{\beta}\right]e^{\beta t}+\dfrac{\alpha}{\beta}$	$\hat{x}(k+1)=\left[x(1)-\dfrac{a}{b}\right]e^{bk}+\dfrac{a}{b}$
Filtering(Prediction)	$dx(t)/dt=[\alpha-\beta dx(0)/dt]e^{\beta t}$	$\Delta\hat{x}(k+1)=\hat{x}(k+1)-\hat{x}(k)$

A fundamental note is made here that the original observations are treated as the approximated derivatives of the dynamic law $x(t)$, however, after the rule finding, the modelling is still required to return back to the derivative level because that is the observational one.

V. PARTIAL DIFFERENTIAL EQUATION MODEL

It is often the case that a variable (or a group of variables, i.e., vector) under investigation relates to multi-factors and the functional relationships are specified by a system of partial equations. Similar to DEMR modelling cases, we may also face the sparse data availability. Therefore, it is necessary to investigate the partial differential equation (system) motivated (multivariate) regression (abbreviated as PDMR) modelling. As a necessary, let us review the partial differential equation (system) theory.

A. A family of partial equation model

The family of partial differential equation system under investigation takes its form:

$$\begin{cases} \dfrac{\partial \underline{z}}{\partial x_i}=\underline{f_i}(\underline{z},\underline{x}), i=1,2,\cdots,m \\ \underline{z}(\underline{x}^{0})=\underline{z}^{0} \end{cases} \qquad (5)$$

where,

$$\underline{z} = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_p \end{bmatrix} \quad \underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} \tag{6}$$

and

$$\underline{f_i}(\underline{z}, \underline{x}) = (f_{i1}(\underline{z}, \underline{x}), f_{i2}(\underline{z}, \underline{x}), \cdots, f_{im}(\underline{z}, \underline{x}))^{\mathrm{T}} \tag{7}$$

B. A Linear Partial Equation System Model

A linear partial differential equation system takes its form:

$$\begin{cases} \dfrac{\partial \underline{z}}{\partial x_i} = A_i(\underline{x})\underline{z} + \underline{b}(\underline{x}), i = 1,2,\cdots,n \\ \underline{z}(\underline{x}^0) = \underline{z}^0 \end{cases} \tag{8}$$

where,

$$A_i(\underline{x}) = (a_{i,jk}(\underline{x}))_{m'm} \tag{9}$$

The solution to a partial differential equation system is not necessary to exist. The following consistent theorem is a necessary condition for a partial equation system to have a solution.

Theorem 1: Frobenius (1877) Assume that functions $\underline{f_i}(\underline{z}, \underline{x})$ are continuously differentiable with respect to x and z respectively in a domain $G \subset \mathbb{R}^m \times \mathbb{R}^n$. Then the equation system has a solution for arbitrary initial data if and only if the following consistency conditions are satisfied

$$\frac{\partial \underline{f_i}}{\partial x_j} + \frac{\partial \underline{f_i}}{\partial \underline{z}} \underline{f_j} = \frac{\partial \underline{f_j}}{\partial x_i} + \frac{\partial \underline{f_j}}{\partial \underline{z}} \underline{f_i}, \forall i \neq j = 1,2,\cdots,n \tag{10}$$

In addition, the solution is unique on the domain where it is defined. In the linear case, the solution is defined on the whole domain $D \subset \mathbb{R}^n$, where the coefficients and free terms are defined, provided the domain is surface-simply connected.

Corollary 2: For a linear partial differential equation system, the consistency conditions can be stated as:

$$\begin{cases} A_i A_j + \dfrac{\partial}{\partial x_j} A_i = A_j A_i + \dfrac{\partial}{\partial x_i} A_j \quad \text{(a)} \\ A_i \underline{b_j} + \dfrac{\partial}{\partial x_j} \underline{b_i} = A_j \underline{b_i} + \dfrac{\partial}{\partial x_i} \underline{b_j} \quad \text{(b)} \end{cases} \tag{11}$$

where $i \neq j = 1,2,\cdots,n$.

C. The consistency conditions for a bivariate PDE model

Let a bivariate PDE takes the form:

$$\begin{cases} \dfrac{\partial z}{\partial x} = \alpha_1(x,y)z + \beta_1(x,y) \\ \dfrac{\partial z}{\partial y} = \alpha_2(x,y)z + \beta_2(x,y) \end{cases} \tag{12}$$

Now let us investigate the formation of Eq. (12) satisfying the Corollary to Forbenius Theorem. Note that:

$$\begin{aligned} A_i &= \alpha_1(x,y) \\ A_j &= \alpha_2(x,y) \\ b_i &= \beta_1(x,y) \\ b_j &= \beta_2(x,y) \\ x_i &= x \\ x_j &= y \end{aligned} \tag{13}$$

For condition ($11i$):

$$\begin{aligned} &A_i A_j + \frac{\partial}{\partial x_j} A_i = A_j A_i + \frac{\partial}{\partial x_i} A_j \\ &\text{LHS}: \alpha_1(x,y)\alpha_2(x,y) + \frac{\partial}{\partial y}\alpha_1(x,y) \\ &\text{RHS}: \alpha_2(x,y)\alpha_1(x,y) + \frac{\partial}{\partial x}\alpha_2(x,y) \end{aligned} \tag{14}$$

which leads to condition:

$$\frac{\partial}{\partial y}\alpha_1(x,y) = \frac{\partial}{\partial x}\alpha_2(x,y) \tag{15}$$

As to condition ($11ii$):

$$\begin{aligned} &A_i \underline{b_j} + \frac{\partial}{\partial x_j} \underline{b_i} = A_j \underline{b_i} + \frac{\partial}{\partial x_i} \underline{b_j} \\ &\text{LHS}: \alpha_1(x,y)\beta_2(x,y) + \frac{\partial}{\partial y}\beta_1(x,y) \\ &\text{RHS}: \alpha_2(x,y)\beta_1(x,y) + \frac{\partial}{\partial x}\beta_2(x,y) \end{aligned} \tag{16}$$

which leads to a fairly complicated condition:

$$\alpha_1(x,y)\beta_2(x,y) + \frac{\partial}{\partial y}\beta_1(x,y) = \alpha_2(x,y)\beta_1(x,y) + \frac{\partial}{\partial x}\beta_2(x,y) \tag{17}$$

Combine Eq. (15) and Eq. (17) together, the consistency conditions can be expressed by:

$$\begin{cases} \dfrac{\partial}{\partial y}\alpha_1(x,y) = \dfrac{\partial}{\partial x}\alpha_2(x,y) \\ \alpha_1(x,y)\beta_2(x,y) + \dfrac{\partial}{\partial y}\beta_1(x,y) = \alpha_2(x,y)\beta_1(x,y) + \dfrac{\partial}{\partial x}\beta_2(x,y) \end{cases} \tag{18}$$

VI. THE PDEMR MODEL FORMATION

Similar to DEMR model, PDEMR model is also constituted by two components: motivated partial differential equation (abbreviated as PDE) systems and coupled (multivariate) regression model. Let us use the linear PDE motivated regression for the basic definition.

Definition 1: Coupled equation system:

$$\begin{cases} \dfrac{\partial z}{\partial x_i} = A_i(\underline{x})\underline{z} + \underline{b}(\underline{x}),\ i = 1, 2, \cdots, m \\ \underline{z}(\underline{x}^0) = \underline{z}^0 \\ D^{\partial}_{x_i(k_i)}\underline{z} = A_i(\underline{x}(\underline{k_i}))\underline{z}(\underline{k_i}) + \underline{b}(\underline{x}(\underline{k_i})) \end{cases} \tag{19}$$

where:

$$\Delta^{\partial}_{x_i(k_i)}\underline{z} = \underline{z}(x_1(k_1), x_2(k_2), \cdots, x_i(k_i), \cdots, x_m(k_m)) - \underline{z}(x_1(k_1), x_2(k_2), \cdots, x_i(k_i - 1), \cdots, x_m(k_m)) \tag{20}$$

denotes the (first) partial difference of $\underline{z}(x_1, x_2, \cdots, x_m)$ with respect to exploratory variable x_i at point $(x_1(k_1), x_2(k_2), \cdots, x_i(k_i), \cdots, x_m(k_m))$.

VII. A PDEMR MODELLING OF PROTEA FREQUENCY COUNT SPATIAL DISTRIBUTION

A. A bivariate partial differential equation for log-count

Bear in mind that we intend to develop a counting model for filling those sites where the counts of a particular class was recorded as zero, typically is in the design note for observation and sampling data collection, however, was not attended for some technical reasons. The count z is a (integer) scalar function of coordinate (x, y) and thus it may be appropriate to us the log-transformation, i.e., $u(x, y)=\ln z(x, y)$. Note that:

$$\begin{cases} \dfrac{\partial u(x,y)}{\partial x} = \dfrac{1}{z(x,y)}\dfrac{\partial z(x,y)}{\partial x} \\ \dfrac{\partial u(x,y)}{\partial y} = \dfrac{1}{z(x,y)}\dfrac{\partial z(x,y)}{\partial y} \end{cases} \tag{21}$$

To obtain the insight of bivariate PDE model, we start with a bivariate partial differential equation system in the form of Eq. (22):

$$\begin{cases} \dfrac{\partial u}{\partial x} = \alpha_1 + 2\alpha_3 x + \alpha_4 y \\ \dfrac{\partial u}{\partial y} = \alpha_2 + \alpha_4 x + 2\alpha_5 y \end{cases} \tag{22}$$

It is obvious that:

$$\begin{aligned} A_i(x, y) &= \alpha_1 + 2\alpha_3 x + \alpha_4 y \\ A_j(x, y) &= \alpha_2 + \alpha_4 x + 2\alpha_5 y \\ \beta_1(x, y) &= 0 \\ \beta_2(x, y) &= 0 \end{aligned} \tag{23}$$

Accordingly, a homogeneous equation system is obtained and it is easy to check that the homogenous equation system Eq. (22) satisfies the consistency conditions set up in Corollary 5.2.2.

The matrix form of Eq. (22) can be written as:

$$\begin{bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} \end{bmatrix} = \begin{bmatrix} \alpha_1 & 2\alpha_3 & \alpha_4 \\ \alpha_2 & \alpha_4 & 2\alpha_5 \end{bmatrix} \begin{bmatrix} 1 \\ x \\ y \end{bmatrix} \tag{24}$$

or taking the transpose for both sides of Eq. (24),

$$\begin{bmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \end{bmatrix} = \begin{bmatrix} 1 & x & y \end{bmatrix} \begin{bmatrix} \alpha_1 & \alpha_2 \\ 2\alpha_3 & \alpha_4 \\ \alpha_4 & 2\alpha_5 \end{bmatrix} \tag{25}$$

Let the parameter matrix be:

$$L = \begin{bmatrix} \alpha_1 & \alpha_2 \\ 2\alpha_3 & \alpha_4 \\ \alpha_4 & 2\alpha_5 \end{bmatrix} \tag{26}$$

the design matrix (a vector) is denoted as:

$$\underline{X} = \begin{bmatrix} 1 \\ x \\ y \end{bmatrix} \tag{27}$$

And the partial derivative vector is denoted as:

$$\frac{\partial u}{\partial \underline{x}} = \begin{bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} \end{bmatrix} \tag{28}$$

Finally we have a matrix representation of the bivariate partial equation system Eq. (22):

$$\left(\frac{\partial u}{\partial \underline{x}}\right)^{\mathrm{T}} = \underline{X}^{\mathrm{T}}\Lambda \tag{29}$$

B. The divided difference and its application in approximating partial derivatives

The key step for PDEMR model setting is the translation from partial derivatives into partial differences. It is often the case that the observations are not equal-gap taken, but on the contrary. In bivariate circumstances, the way for defining difference for unequal-gapped data is even more complicated than that in one-dimensional case. Therefore, we intend to develop a scheme of the obtaining "best" partial difference for approximating the corresponding partial derivatives.

(i) Divided difference

Definition 1: Given a function $f(x)$ on the interval $[a,b]$. Let the sequence $\{x_1, x_2, \cdots, x_l\}$ with $\forall x_i \in [a,b]$ and $x_i < x_j$ for any $i<j$ Then the quantity:

$$\Delta^{\partial}_{x_i} f \underset{=}{\Delta} \frac{f(x_i)-f(x_{i-1})}{x_i - x_{i-1}} \tag{30}$$

is called the (first) divided difference for function $f(\cdot)$ at x_i.

(ii) Partial divided difference

Definition 2: Given a bivariate function $w(x, y)$ on $D \subset \mathbb{R}^2$. Let $(x_i, y_j) \in D$, then:

$$\Delta^{\partial}_{x_i} w = \frac{w(x_i, y_j) - w(x_{i-1}, y_j)}{x_i - x_{i-1}} \tag{31}$$

is defined as a (first) partial difference of $w(\cdot,\cdot)$ with respect to exploratory variable x at (x_i, y_j). Similarly,

$$\Delta^{\partial}_{y_j} w = \frac{w(x_i, y_j) - w(x_i, y_{j-1})}{y_j - y_{j-1}} \tag{32}$$

is defined as the partial difference of $w(\cdot,\cdot)$ with respect to exploratory variable y at (x_i, y_j).

(iii) Using directional derivative for least-square estimated partial divided difference

Let D be a sub-space of a 2-dimensional space, $\mathbb{R}\times\mathbb{R}$, any point of D, denoted as $M(x, y)$ corresponds to the value of a scalar function $s(x, y)$, if the position of M could be represented by a vector $\boldsymbol{r}$, then scalar function can be regarded as a function of variable vector $\boldsymbol{r}$, i.e., $s = s(\boldsymbol{r})$.

Definition 3: Let:

$$\text{grad}\, s = \nabla s \underset{=}{\Delta} \frac{\partial s}{\partial x}\boldsymbol{i} + \frac{\partial s}{\partial y}\boldsymbol{j} \tag{33}$$

be the gradient of scalar field $s(x,y)$ at point (x,y). Let $\boldsymbol{l}$ be a unit directional vector with directional angular θ_x and θ_y such that:

$$|l| = \cos^2(\theta_x) + \cos^2(\theta_y) = 1 \tag{34}$$

Then:

$$\frac{\partial s}{\partial l} = \boldsymbol{l} \cdot \text{grad}\, s = \frac{\partial s}{\partial x}\cos(\theta_x) + \frac{\partial s}{\partial y}\cos(\theta_y) \tag{35}$$

is called the directional derivative with respect to directional vector $\boldsymbol{l}$ at point (x, y).

Let $\mathbb{k}_r(x, y) \subset D$ be a neighborhood of radius r, i.e., for any $(x_i, y_j) \in \mathbb{k}_r(x, y)$ the distances of (x_i, y_j) from point

$$(x, y) : \sqrt{(x_i - x)^2 + (y_j - y)^2} < r.$$

However, unless the functional form of the scalar field is available, then we can not obtain the accurate values of the directional derivatives. However, for each direction, $(x, y) \to (x_i, y_j)$, an approximate directional derivative can be calculated as:

$$\frac{\widehat{\partial \varphi}}{\partial l} = \frac{s(x,y) - s(x_i, y_j)}{\sqrt{(x-x_i)^2 + (y-y_j)^2}} \tag{36}$$

Furthermore, the cosines of the directional angular are also calculated as:

$$\begin{cases} \cos(\theta_x) = \dfrac{x - x_i}{\sqrt{(x-x_i)^2 + (y-y_j)^2}} \\ \cos(\theta_y) = \dfrac{y - y_j}{\sqrt{(x-x_i)^2 + (y-y_j)^2}} \end{cases} \tag{37}$$

Therefore, the $(x, y) \to (x_i, y_j)$ pair of point will generate an equation:

$$\Delta^{\partial}_x s \cos(\theta_x) + \Delta^{\partial}_y s \cos(\theta_y) = \left(\frac{\widehat{\partial \varphi}}{\partial l}\right)^{(x_i, y_j)}_{(x,y)} \tag{38}$$

In general, there will be $k(k-1)/2$ equations in total if there are k points in $k_r(x, y) \subset D$, which overspecify the two unknown partial differences, Δ^{∂}_x and Δ^{∂}_y at (x, y) respectively. As a matter of fact, the partial differences will be least-square estimate.

C. The coupled bivariate regression model

Once the partial differences, either direct divided estimate or the least-square estimate defined in B, Δ^{∂}_x and Δ^{∂}_y are ready for further analysis, let:

$$\Delta_{n\times 2} = \begin{bmatrix} \Delta^{\partial}_{x_1} u & \Delta^{\partial}_{y_1} u \\ \Delta^{\partial}_{x_2} u & \Delta^{\partial}_{y_2} u \\ \vdots & \vdots \\ \Delta^{\partial}_{x_n} u & \Delta^{\partial}_{y_n} u \end{bmatrix} \quad X_{n\times 3} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ \vdots & \vdots & \vdots \\ 1 & x_n & y_n \end{bmatrix}$$
$$\Lambda_{3\times 2} = \begin{bmatrix} \alpha_1 & \alpha_2 \\ 2\alpha_3 & \alpha_4 \\ \alpha_4 & 2\alpha_5 \end{bmatrix} \quad E_{n\times 2} = \begin{bmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \\ \vdots & \vdots \\ e_{n1} & e_{n2} \end{bmatrix} \tag{39}$$

Then the coupled regression model in matrix form will be:

$$\Delta = X\Lambda + E \tag{40}$$

Finally, the bivariate PDEMR model for the log-count will be:

$$\begin{cases} \begin{bmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \end{bmatrix} = [1 \;\; x \;\; y] \begin{bmatrix} \alpha_1 & \alpha_2 \\ 2\alpha_3 & \alpha_4 \\ \alpha_4 & 2\alpha_5 \end{bmatrix} \\ \Delta = X\Lambda + E \end{cases} \tag{41}$$

As to the error structure of the PDEMR formation in Eq. (42), the error $e_{ij} \sim N(\xi_i, \sigma_i^2), i = 1, 2, \cdots, n;\, j = 1, 2$ are assumed to be independent normal random fuzzy variables. For details, see Liu (2004, 2007).

VIII. PDEMR PREDICTED PROTEA FREQUENCY COUNTS

Using the predicted results from the PDEMR model, the unsampled cells are predicted with frequency counts of the Protea. Figure 5 shows the PDEMR model predicted frequency counts of Proteas in the population size of 1 to 10, in the Cape Floristic Region, from 1992 to 2002. The figures show a general range of predicted values, but in fact the actual predicted values are numerical, and have predicted the sample values of where there were zero samples.

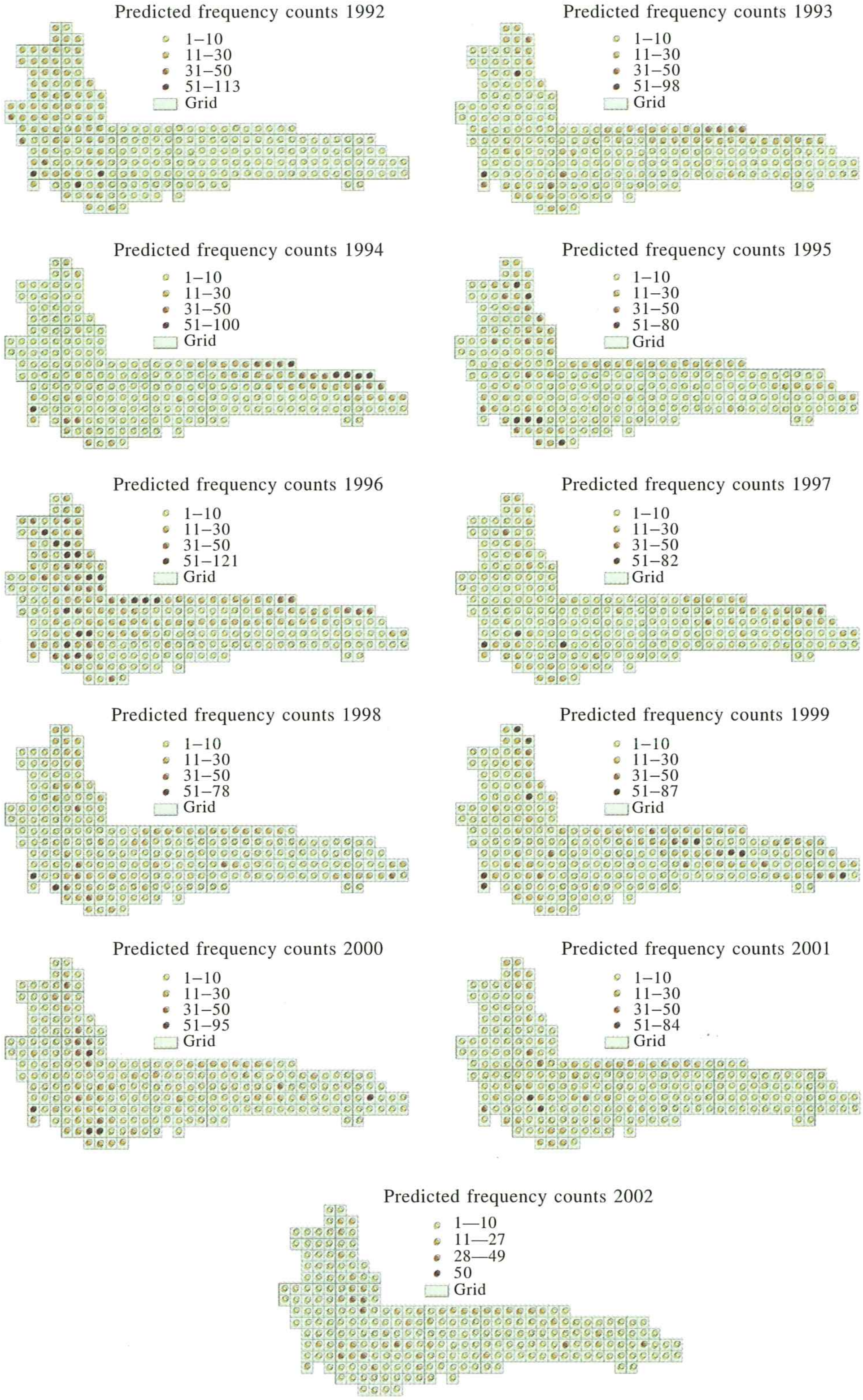


Figure 5. The PDEMR model predicted frequency counts of proteas in the population size of 1–10, in the cape floristic region, 1992–2002

Finally, we can produce kriging prediction maps of the Proteas, using the predicted results from the PDEMR model. Figure 6 shows the distribution and patterns of frequency counts of Proteas. One can see the changes in the density of occurrence of the Proteas in the Cape Floristic Region over the 11 years.

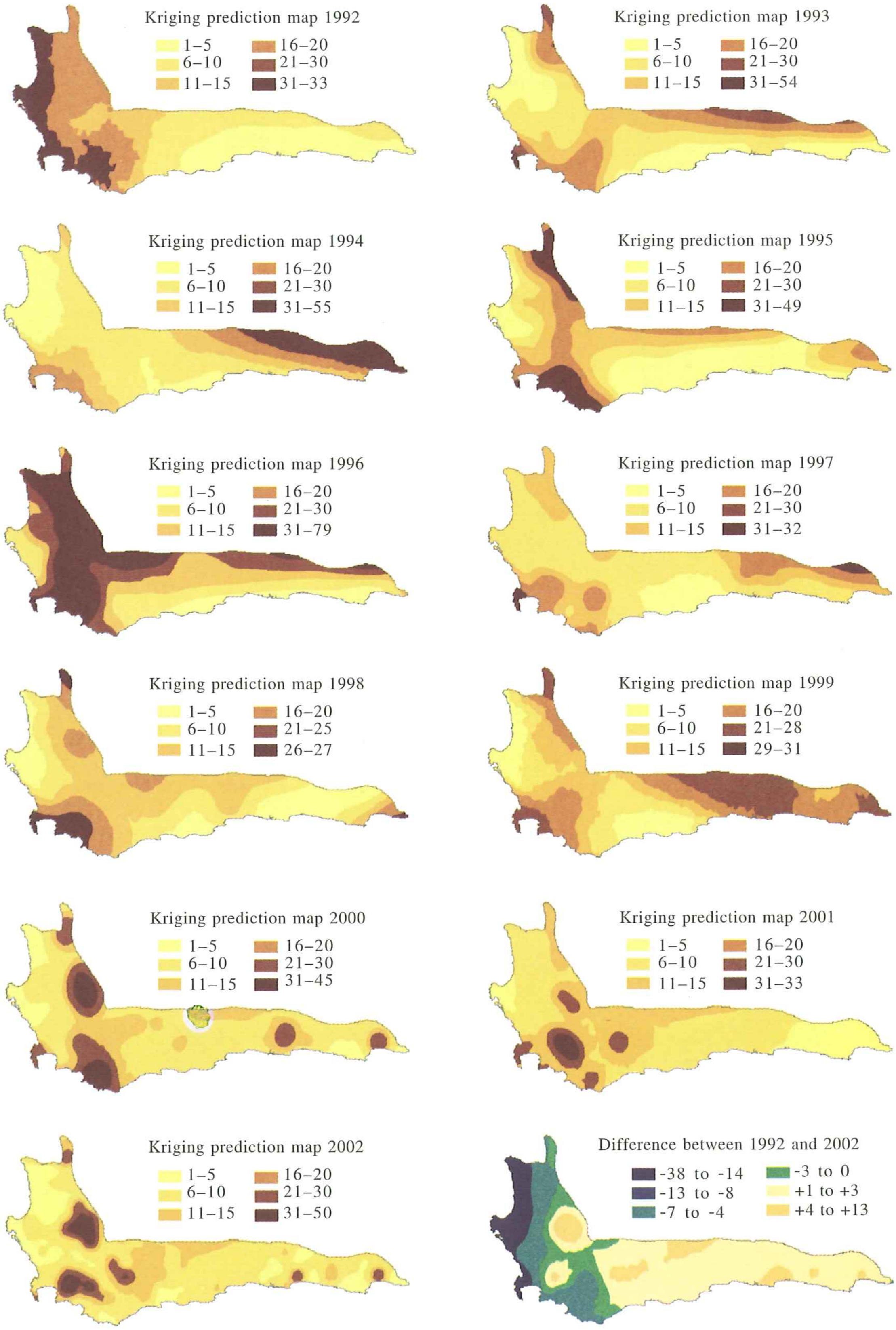


Figure 6. The kriging prediction maps of frequency counts of proteas in the population size of 1–10, in the cape floristic region, 1992–2002

One must recognise that this category of Proteas only has the population size of 1 to 10, so that it is very sensitive to environmental changes. Therefore, the Proteas between the 11 years have changed dramatically in frequency count and spatial distribution patterns. It is however clear that in 1992 there were more high frequency count areas, and in 2002 there were much less high frequency count areas. This could indicate habitat fragmentation for the Protea species. The final map in Figure 6 shows the difference between 1992 and 2002, one can see that there has been some high magnitude negative changes as shown by blue and green colours, and while the positive changes covers a large area but the changes are smaller in value as shown by the orange colour. This could indicate that under the changing climatic conditions, the Protea species is expanding its habitat, but only by a increase in count, 1 to 13 over some areas. However, on the west coast, large areas of Proteas are decreasing in numbers.

IX. CONCLUSION

In this paper, we solved two crucial problems with regard to the environmental dataset, presence data only and incomplete sample data. We used the *partial differential equation motivated regression* (PDEMR) model, which merges the partial differential equation theory, (statistical) linear model theory and credibility measure theory together. The coupled regression component in a PDEMR model is in nature a special random fuzzy multivariate regression model. We developed a bivariate model for prediction of the Protea species in the population size of 1 to 10, in the Cape Floristic Region, 1992 to 2002, in South Africa. The model has produced very good results, which helped to produce kriging prediction maps. The spatial distribution and pattern are clear to see and understand in the kriging maps.

ACKNOWLEDGEMENTS

We would like to thank the Protea Atlas Project for providing the species data for our research.

REFERENCES

[1] Deng J L., 1985, *Grey Systems (Social Economical)*, The Publishing House of Defense Industry (in Chinese), Beijing, China.

[2] Freeth R., Bomhard B., Midgley G., 2007, Adapting to Climate Change in the Cape Floristic Region: Building Resilience of People and Plants in Protected Areas. South African National Biodiversity Institute (SANBI), Climate Change Research Group, Kirstenbosch Research Center.

[3] Frobenius G., 1877, *Ueber das Pfaffsche problem*, J. Reine Angew. Math., 82: 230–315.

[4] Guo D., Guo R., Midgley G F., Ebrahim I., 2007, DEMR Modelling of Brunsvigia Litoralis Plant Distribution in Western Region District Council, South Africa , *Proceedings of the Sixth International Conference on Information and Management Sciences*, Lhsa, Tibet, China, July 1–6.

[5] Guo D., Guo R., Thiart C., 2006, The Coupling of regression Modelling and Differential Equation Model in GM(1,1) Modelling and Extended GM(1,1) Models, *Journal of Grey System*, 9(2): 143–154.

[6] Guo R., 2005, Repairable System Modelling via Grey Differential Equations, *Journal of Grey System*, 8(1): 69–92.

[7] Guo R., Guo D., Midgley G F., Rebelo A G., 2007, Bivariate DEMR Modelling of Range Shifts in Aulax Umbellata Shrubs From 1992 to 2002 in South Africa, in *Proc. 6th International Conference on Information and Management Sciences*, Lhasa, Tibet, 783–791.

[8] Hakopian H A., Tonoyan M G., 2004, Partial differential analogs of ordinary differential equations and systems. New York Journal of Mathematics, 10: 89–116

[9] Liu B D., 2004, Uncertainty Theory: An Introduction to Its Axiomatic Foundations. Springer-Verlag Heidelberg, Berlin.

[10] Liu B D., 2007, *Uncertainty Theory*, 2nd Edition. Berlin, Springer-Verlag Heidelberg.

[11] Liu Y K., Liu B D., 2002, Random Fuzzy Programming with Chance Measures Defined by Fuzzy Integrals, Mathematical and Computer Modelling, 2(3): 249–256.

[12] Midgley G F., Hughes G O., Thuiller W., Rebelo A G., 2006, Migration rate limitations on climate change-induced range shifts in Cape Proteaceae, *Diversity and Distributions*, 12: 555–562.

[13] Rebelo A G., 2001, *Proteas. A field guide to the proteas of southern Africa*. Fernwood Press, Vlaeberg, Cape Town, South Africa.

[14] Zadeh L A., 1965, Fuzzy Sets, *Information and Control*, 8: 338–353.

[15] Zadeh L A., 1978, Fuzzy Sets as a Basis for a Theory of Possibility, *Fuzzy Sets and Systems*, 1: 3–28.

APPENDIX: THEORY OF RANDOM FUZZY VARIABLE

First we need to review the fuzzy credibility measure theory foundation proposed by Liu (2004), and then state the concept of random fuzzy variable. The theory of Liu (2004, 2007) is different from that initiated by Zadeh (1965, 1978).

Let Θ be a nonempty set, and 2^{Θ} the power set on Θ. Each element, let us say, $A \subset \Theta, A \in 2^{\Theta}$, is called an event. A number denoted as, $\mathrm{Cr}\{A\}, 0 \le \mathrm{Cr}\{A\} \le 1$, is assigned to event $A \in 2^{\Theta}$, which indicates the credibility grade with which event $A \in 2^{\Theta}$ occurs. $\mathrm{Cr}\{A\}$ satisfies following axioms given by Liu (2004):

Axiom 1: $\mathrm{Cr}\{\Theta\}=1$.

Axiom 2: $\mathrm{Cr}\{\cdot\}$ is non-decreasing, i.e., whenever $A \subset B, \mathrm{Cr}\{A\} \le \mathrm{Cr}\{B\}$.

Axiom 3: $\mathrm{Cr}\{\cdot\}$ is self-dual, i.e., for any $A \in 2^{\Theta}$, $\mathrm{Cr}\{A\}+\mathrm{Cr}\{A^c\}=1$.

Axiom 4: $\mathrm{Cr}\{\cup_i A_i\} \wedge 0.5 = \sup_i [\mathrm{Cr}\{A_i\}]$ for any $\{A_i\}$ with $\mathrm{Cr}\{A_i\} \le 0.5$.

Definition A.1: Liu (2004) Any set function $\mathrm{Cr}: 2^{\Theta} \to [0,1]$ satisfies *Axioms 1-4* is called a $(\vee, \wedge)$ -credibility measure (or classical credibility measure). The triple $(Q, 2^{\Theta}, \mathrm{Cr})$ is called the

-credibility measure space.

Definition A.2: Liu (2004) A fuzzy variable ξ is a mapping from credibility space $(Q, 2^{\Theta}, \text{Cr})$ to the set of real numbers, i.e.,

$\xi : (\Theta, 2^{\Theta}, \text{Cr}) \rightarrow \mathbb{R}$.

Definition A.3: Liu (2004) The credibility distribution $\Phi : \mathbb{R} \rightarrow [0,1]$ of a fuzzy variable ξ on $(Q, 2^{\Theta}, \text{Cr})$ is:

$$\Phi(x) = \text{Cr}\{\theta \in \Theta \mid \xi(\theta) \leq x\} \quad (1)$$

Now we are ready to state the random fuzzy variable concept.

Definition A.4: A random fuzzy variable, denoted as $\xi = \{X_{\beta(\theta)}, \theta \in \Theta\}$, is a collection of random variables X_β defined on the common probability space $\{\Omega, \mathfrak{A}, Pr\}$ and indexed by a fuzzy variable $\beta(\theta)$ defined on the credibility space $(Q, 2^{\Theta}, \text{Cr})$.

Definition A.5: Liu (2004) Let x be a random fuzzy variable, then the average chance measure denoted by $\text{ch}\{\times\}$, of a random fuzzy event $\{\xi \leq x\}$, is:

$$\text{ch}\{\xi \leq x\} = \int_0^1 Cr\{\theta \in \Theta \mid \Pr\{x(q) \leq x\}^3 \leq \alpha\} d\alpha \quad (2)$$

Then function $\Upsilon\{\times\}$ is called as average chance distribution if and only if:

$$\Upsilon(x) = \text{ch}\{\xi \leq x\} \quad (3)$$

Now, it is time to find the average chance distribution for a normal random fuzzy variable x^d: $N(z, s^2)$, where ζ is a triangular fuzzy variable and σ^2 is a given positive real number. The fuzzy mean is assumed to have a trapezoidal membership function:

$$m_z(w) = \begin{cases} \frac{w-a}{b-a} & a \leq w < b \\ 1 & b \leq w < c \\ \frac{d-w}{d-c} & c \leq w < d \\ 0 & \text{otherwise} \end{cases} \quad (4)$$

and

$$\Phi(w) = \text{Cr}\{\xi \leq w\} = \begin{cases} 0 & w < a \\ \frac{w-a}{2(b-a)} & a \leq w < b \\ \frac{1}{2} & b \leq w < c \\ \frac{w+d-2c}{2(d-c)} & c \leq w < d \\ 1 & \text{otherwise} \end{cases} \quad (5)$$

which gives the credibility distribution for the fuzzy mean, ζ.

Then the critical step is to derive the expression of $\text{Cr}\{\zeta(\theta) \in \Theta \mid \Pr\{\xi(\omega, \theta) \leq x\}^3 \geq \alpha\}$. For normal random fuzzy variable with a triangular fuzzy mean, note that:

$$\{\zeta(\theta) : \Pr\{\zeta(\omega, \theta) \leq x\} \geq \alpha \\ \Leftrightarrow \{\theta \in \Theta) \leq x - \sigma\Phi^{-1}(\alpha)\} \quad (6)$$

Then the range for the integration of the integrand $\text{Cr}\{\theta \in \Theta : \zeta(\theta) \leq x - \sigma\Phi^{-1}(\alpha)\}$ with respect to is listed in Table A1.

Table A1: Integration range with respect to α, where $\zeta = g(\alpha) = x - \sigma\,\Phi^{-1}(\alpha)$

$g(\alpha)$	Range for α	$\text{Cr}\{\theta \in \Theta : \zeta(\theta) \leq x - \sigma\Phi^{-1}(\alpha)\}$
$-\infty < g(\alpha) < a$	$\Phi\left(\frac{x-a}{\sigma}\right) < \alpha < 1$	0
$a \leq g(\alpha) < b$	$\Phi\left(\frac{x-b}{\sigma}\right) < \alpha < \Phi\left(\frac{x-a}{\sigma}\right)$	$\frac{x - \sigma\Phi^{-1}(\alpha) - a}{2(b-a)}$
$b \leq g(\alpha) < c$	$\Phi\left(\frac{x-c}{\sigma}\right) < \alpha < \Phi\left(\frac{x-b}{\sigma}\right)$	$\frac{1}{2}$
$c \leq g(\alpha) < d$	$\Phi\left(\frac{x-c}{\sigma}\right) < \alpha < \Phi\left(\frac{x-d}{\sigma}\right)$	$\frac{x - \sigma\Phi^{-1}(\alpha) + d - 2c}{2(d-c)}$
$g(\alpha) < d$	$0 < \alpha < \Phi\left(\frac{x-d}{\sigma}\right)$	1

Then we obtain the average chance distribution:

$$\begin{aligned} \Upsilon_\xi &= ch\{\xi(\omega, \theta) \leq x\} \\ &= \frac{x-a}{2(b-a)}\left(\Phi\left(\frac{x-a}{\sigma}\right) - \Phi\left(\frac{x-b}{\sigma}\right)\right) + \\ &\frac{x+d-2c}{2(d-c)}\left(\Phi\left(\frac{x-c}{\sigma}\right) - \Phi\left(\frac{x-d}{\sigma}\right)\right) + \\ &\frac{1}{2}\left(\Phi\left(\frac{x-b}{\sigma}\right) - \Phi\left(\frac{x-c}{\sigma}\right)\right) + \Phi\left(\frac{x-d_v}{\sigma}\right) - \\ &\frac{\sigma}{2(b-a)} \int_{\frac{x-b}{\sigma}}^{\frac{x-a}{\sigma}} u\phi(u)du - \frac{\sigma}{2(d-c)} \int_{\frac{x-d}{\sigma}}^{\frac{x-c}{\sigma}} u\phi(u)du \end{aligned} \quad (7)$$