# Valuation of stock loans with exponential phase-type Lévy models

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**Abstract**

Stock loans are collateral loans with stocks used as the collateral. This paper is concerned with a stock loan valuation problem in which the underlying stock price is modeled as an exponential phase-type Lévy model. The valuation problem is formulated as the optimal stopping problem of a perpetual American option with a time-varying exercise price. When a transformation is applied to the perpetual American option, it becomes a perpetual American call option in an economy with a negative interest rate, thus causing standard Wiener-Hopf techniques to fail. We solve this optimal stopping problem using a variational inequality approach.

*Key words:* Stock loans, Phase-type Lévy models, Optimal stopping.

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## 1. Introduction

A stock loan, a type of equity securities lending service, is a loan that is collateralized with stocks and issued by a financial institution (the lender) to a client (the borrower). The size of the securities lending market reached its peak at nearly US$850 billion in 2007. After short-selling restrictions were imposed on the U.S. securities market in 2008, the value of U.S. equities on loan was still nearly US$250 billion; see Standard & Poor's (2009). This huge value of stock loans transactions has stimulated interest in the appropriate valuation of these loans in a general market situation.

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A stock loan contract grants the borrower the right to repay the loan at any time or simply to default on it and lose the collateral. The borrower's early redemption right can be regarded as a perpetual American option, that is, the borrower can exercise the option at any time without a time limit. The value of this perpetual American option is therefore of central importance to the problem of stock loan valuation.

That value can be expressed as an ordinary perpetual American call option with a negative (effective) interest rate, which creates the major challenge of stock loan pricing. Consider the case of geometric Brownian motion (GBM) for the stock price. The optimal exercise rule of a perpetual American call option is to exercise the option the first time the stock price rises and crosses a constant level, that is, the constant optimal exercise boundary. If the interest rate is positive, then the stock price will cross any fixed boundary almost surely. If the interest rate is negative, in contrast, the problem becomes complicated. Given any fixed boundary level greater than the current stock price, there is a positive probability that the stock price will never cross that level.

Xia and Zhou (2007) are pioneers in solving the stock loan problem. They valued stock loans using the classical GBM model and a purely probabilistic approach. Zhang and Zhou (2009) then extended their framework to a regime switching model and solved the problem using variational inequalities. Dai and Xu (2009) investigated the stock loan problem with finite maturity under GBM, and Yam et al. (2010) considered callable stock loans.

Although most studies on the stock loan valuation problem adopt the GBM approach for the underlying stock price, empirical evidence (e.g., Andersen et al., 2002; Pan, 2002; Eraker et al., 2003) shows that the jump diffusion model would be a better asset price model for capturing the heavy tails of the empirical distribution. A jump diffusion model with a flexible jump distribution is therefore worth considering for stock loan valuation.

Merton (1976) proposes a jump diffusion model for option pricing that employs a Gaussian distributed jump size. Another notable jump diffusion model is the double-exponential jump diffusion model proposed by Kou (2002). The generalization of the jump diffusion model is an exponential Lévy model, such as the variance-gamma model (Madan et al., 1998), CGMY model (Carr et al., 1999) or normal inverse Gaussian model (Barndorff-Nielssen, 2000).

Sun (2010) recently considered the stock loan valuation problem under the framework of the double-exponential jump diffusion model. Although this constitutes a good start, the asset return distribution is not sufficiently flexible to capture the empirical distribution implied by market data. For this reason, we here consider phase-type jump diffusion for stock loan valuation.

Phase-type distribution is dense over the class of all positive valued distributions. By making use of this fact, Asmussen et al. (2007) show that the class of phase-type jump diffusion models is dense over all exponential Lévy models. In other words, the option price derived from phase-type jump diffusion models can be used to approximate the corresponding price under a general exponential Lévy model. Asmussen et al. (2007) approximate the CGMY model by phase-type jump diffusion. In fact, the phase-type jump diffusion model embraces both the Kou (2002) model and the mixed-exponential jump diffusion model (Cai and Kou, 2011) as special cases.

Asmussen et al. (2004) solved the price of the perpetual American put option with a positive interest rate using phase-type jump diffusion models. They employed the Wiener-Hopf factorization technique proposed in Mordecki (2002) to derive the optimal exercise boundary. The pricing problem can then be converted into the evaluation of an expectation at the given exercise boundary.

Although Wiener-Hopf factorization is useful in solving American option pricing problems involving Lévy processes, particularly, phase-type Lévy models, it relies heavily on the assumption of a positive interest rate or, in the limiting case, zero interest rate. The method is not applicable to a negative effective interest rate in the stock loan pricing problem. In this paper, we adopt the variational inequality approach for American option pricing (Zhang and Zhou, 2009; Israel and Rincon, 2008).

Using the phase-type jump diffusion model, we show that the price of the perpetual American option satisfies an ordinary integro-differential equation (OIDE). The solution of this OIDE is closely linked to the root characteristics of a Cramér-Lundberg equation (C-L equation). The root characteristics of the C-L equation are first investigated in a special case in which the random jump size follows a hyperexponential distribution, a special phase-type distribution. By making use of this special case, we then construct a novel transformation to extend the result to a fairly general class of phase-type Lévy models. The derived analytical pricing formula is not only useful in pricing stock loans, but also in pricing a traditional perpetual American call option on a dividend-paying stock, which follows the phase-type Lévy model where the dividend yield is greater than the interest rate.

The remainder of the paper is organized as follows. Section 2 introduces the model and the stock loan valuation problem. Section 3 presents several properties of stock loans in general phase-type Lévy models, and Section 4 derives the analytical formulas of stock loans in a fairly general phase-type Lévy model. Section 5 concludes the paper.

## 2. Problem formulation

This section presents the formulation of stock loan valuation in a phase-type Lévy model. We begin by introducing a phase-type jump-diffusion model for the underlying stock price.

### 2.1. The stock price process

Phase-type distributions have many applications in queuing theory, insurance and ruin probability. The classical book by Asmussen (2000) contains detailed information in this area. We extract a number of important properties that are useful in the present paper. Consider a continuous-time Markov process with one transient state and one absorption state. The intensity matrix is given by

$$\begin{pmatrix} -\theta & \theta \\ 0 & 0 \end{pmatrix},$$

where $\theta > 0$. If $Y$ is the absorption time of this Markov process, then it follows an exponential distribution. The cumulative distribution function is

$$F_Y(y) = 1 - e^{-\theta y}. \tag{1}$$

A finite mixture of exponential distribution is called hyperexponential distribution, which can be expressed as the absorption time of a continuous-time Markov process with $m$ transient states and one absorption state with an intensity matrix of the form

$$\begin{pmatrix} -\theta_1 & \cdots & 0 & \theta_1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & -\theta_m & \theta_m \\ 0 & \cdots & 0 & 0 \end{pmatrix}.$$

The cumulative distribution function becomes

$$F_Y(y) = \sum_{i=1}^{m} \alpha_i \left(1 - e^{-\theta_i y}\right), \tag{2}$$

where $\alpha_i \geq 0$ and $\sum_{i=1}^{m} \alpha_i = 1$. $\alpha_i$ represents the probability of the process starting at state $i$. Using matrix notation,

$$F_Y(y) = 1 - \boldsymbol{\alpha} e^{\mathbf{T} y} \mathbf{1}, \tag{3}$$

where $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m)$, $\mathbf{T} = \text{diag}(-\theta_1, \ldots, -\theta_m)$, and $\mathbf{1} = (1, \ldots, 1)^T$.

Generalizing this concept, phase-type distribution describes the absorption time of a finite state continuous-time Markov process with $m$ transient states and one absorption state. Let $\mathbf{T}$ be the intensity matrix of the transient states and $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m)$ be an initial probability vector. Phase-type distribution is parametrized by $(m, \mathbf{T}, \boldsymbol{\alpha})$. The full intensity matrix of the Markov process can be written as

$$\boldsymbol{S} = \begin{pmatrix} \mathbf{T} & \mathbf{t} \\ \mathbf{0} & 0 \end{pmatrix},$$

where $\mathbf{t} = -\mathbf{T}\mathbf{1}$. The cumulative distribution function is

$$F_Y(y) = 1 - \boldsymbol{\alpha} e^{\mathbf{T}y}\mathbf{1}, \tag{4}$$

the density function is

$$f_Y(y) = \boldsymbol{\alpha} e^{\mathbf{T}y}\mathbf{t}, \tag{5}$$

and the moment generating function is

$$M(t) = \mathbb{E}[e^{tY}] = \boldsymbol{\alpha}(-t\mathbf{I} - \mathbf{T})^{-1}\mathbf{t}. \tag{6}$$

Phase-type distributions constitute a very rich class. As shown in Johnson and Taafe (1988), phase-type distribution is dense in the field of all distributions on $(0, \infty)$. When $\mathbf{T}$ is a diagonal matrix, this distribution is reduced to a hyperexponential distribution.

A common application of the phase-type distribution is the modeling of interclaim arrival time (e.g., Asmussen, 2000; Song et al., 2010). Asmussen et al. (2004) introduced phase-type distribution to model jumps in stock prices. We follow their model, in which the stock price process, $\{S_t\}$, defined on a risk-neutral probability space $\left(\Omega, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{F}\right)$ is given by

$$\begin{aligned} S_t &= \exp(X_t), \qquad (7)\\ X_t &= x + \mu t + \sigma W_t + \sum_{i=1}^{N_t} Y_i, \end{aligned}$$

where $\mu = r - \sigma^2/2 - \lambda\left(\mathbb{E}(e^{Y_i}) - 1\right)$. The constant parameters $r$ and $\sigma$ are the instantaneous interest rate and the volatility of the stock, respectively. The stochastic process $\{N_t\}$ represents a Poisson process with constant intensity $\lambda$, and the jump size $Y_i$, $i \in \mathbb{N}$, follows a two-sided phase-type distribution with the density function,

$$f_Y(y) = p\boldsymbol{\alpha}^+ e^{\mathbf{T}^+ y}\mathbf{t}^+ I_{\{y\geq 0\}} + (1-p)\boldsymbol{\alpha}^- e^{-\mathbf{T}^- y}\mathbf{t}^- I_{\{y<0\}}. \tag{8}$$

As the jump diffusion financial market is incomplete, not all of the contingent claims can be perfectly hedged, and there are infinitely many equivalent martingale measures. Our choice of martingale measure, $\mathbb{P}$, is the one that preserves the phase-type structure of the log-price $X_t$, as proposed by Asmussen et al. (2004).

### 2.2. *Stock loans*

Stock loans are collateral loans in which stocks are used as collateral. The borrower receives the loan principal ($q$), pays the service charge ($c$), and has the right to repay the principal with interest (continuously compounded at rate $\gamma$) and regain the stock at any future time. These transactions can be summarized as follows.

- The borrower receives a cash amount of $q-c$ and $V_0$, a perpetual American option with time-varying strike price $qe^{\gamma t}$.
- The bank receives $S_0$ (one unit of stock) as collateral.

By equating the benefits of both parties, the service charge is deduced as

$$c = q + V_0 - S_0. \tag{9}$$

The corresponding perpetual American option has the following presentation.

$$V_0 = V(x) = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_0} \mathbb{E}\left[e^{-r\tau}\left(S_\tau - qe^{\gamma\tau}\right)^+ I_{\{\tau<\infty\}} | S_0 = e^x\right], \tag{10}$$

where $\mathcal{T}_u$, $u \geq 0$, is the set of all stopping time taking values in the time interval $(u, \infty)$. By taking the transformation $\widetilde{S}_t = S_t e^{-\gamma t}$, the value can be written as

$$V(x) = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_0} \mathbb{E}\left[e^{-(r-\gamma)\tau}\left(\widetilde{S}_\tau - q\right)^+ I_{\{\tau<\infty\}} | \widetilde{S}_0 = e^x\right], \tag{11}$$

which is the value of a perpetual American option with a constant strike price and a possibly negative effective interest rate, $\widetilde{r} = r - \gamma$.

From now on, we use the transformed stock price process $\widetilde{S}_t$ as the underlying stock of the American option, and define $\widetilde{X}_t$ as the transformed log-price. Their dynamics are given by

$$\widetilde{S}_t = \exp(\widetilde{X}_t), \tag{12}$$

$$\widetilde{X}_t = x + \widetilde{\mu} t + \sigma W_t + \sum_{i=1}^{N_t} Y_i, \tag{13}$$

where $\widetilde{\mu} = r - \gamma - \sigma^2/2 - \lambda\left(\mathbb{E}(e^{Y_i}) - 1\right)$.

## 3. Stock loan properties in a phase-type Lévy model

Several basic properties of the perpetual American option associated with stock loans are useful in deriving the closed-form solution in a phase-type Lévy model. Take $S = e^x$ and write $v(S) = V(\ln S) = V(x)$. Lemmas 3.1 and 3.2 below are the stock loan properties of the underlying stock following a continuous-time Markov process. These results are taken from Xia and Zhou (2007) and the proofs are thus omitted.

**Lemma 3.1.** $v(S)$, a deterministic function of the initial stock price $S$, satisfies the following properties.

1. $(S-q)^+ \le v(S) \le S$ for all $S > 0$.
2. $v(S)$ is convex, continuous and nondecreasing in $S$ on $(0, \infty)$.

**Lemma 3.2.** Define $k = \inf\{S > 0 : S - q \ge v(S)\} \ge q$, where $\inf \emptyset = \infty$. Then, $\{S > 0 : S - q \ge v(S)\} = [k, \infty)$.

**Theorem 3.1.** If $\widetilde{X}_t$ follows a Lévy process, then the optimal stopping time takes the form

$$\tau_b = \inf\left\{t \ge 0 : \widetilde{X}_t \ge b\right\}, \tag{14}$$

where $b$ is a constant.

*Proof.* The stock loan value at time $t$ can be written as

$$\begin{aligned}
V_t &= v(S_t) \\
&= \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_t} \mathbb{E}\left[e^{-r(\tau-t)}\left(S_t e^{X_\tau - X_t} - q e^{\gamma\tau}\right)^+ I_{\{\tau<\infty\}} \mid \mathcal{F}_t\right] \\
&= e^{\gamma t} \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_t} \mathbb{E}\left[e^{-r(\tau-t)}\left(e^{-\gamma t} S_t e^{X_\tau - X_t} - q e^{\gamma(\tau-t)}\right)^+ I_{\{\tau<\infty\}} \mid \mathcal{F}_t\right] \\
&= e^{\gamma t} \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_0} \mathbb{E}\left[e^{-r\tau}\left(x e^{X_\tau} - q e^{\gamma\tau}\right)^+ I_{\{\tau<\infty\}} \mid \mathcal{F}_0\right]_{x = e^{-\gamma t} S_t} \\
&= e^{\gamma t} v(e^{-\gamma t} S_t).
\end{aligned}$$

Hence, the optimal stopping time (cf. Karatzas and Shreve, 1998, Chapter 2.5) is

$$\begin{aligned}\tau^* &= \inf\left\{t \geq 0 : S_t - qe^{\gamma t} \geq v(S_t)\right\} \\ &= \inf\left\{t \geq 0 : S_t - qe^{\gamma t} \geq e^{\gamma t} v(e^{-\gamma t} S_t)\right\} \\ &= \inf\left\{t \geq 0 : S_t e^{-\gamma t} - q \geq v(e^{-\gamma t} S_t)\right\} \\ &= \inf\left\{t \geq 0 : e^{-\gamma t} S_t \geq k\right\} \\ &= \inf\left\{t \geq 0 : \widetilde{X}_t \geq \ln k\right\},\end{aligned}$$

where $k$ is the value defined in Lemma 3.2. □

Theorem 3.1 asserts that the optimal stopping time is a first passage time. It is optimal to exercise the perpetual American option at the first time the transformed log-price exceeds a predetermined level. Such a level is called the optimal exercise boundary. We denote the optimal exercise boundary by $b^*$ and the optimal stopping time by $\tau_{b^*}$. Theorem 3.1 greatly simplifies the optimization problem. The original problem requires a search of all possible stopping times. However, the optimal stopping time is a first passage time, and hence we only need to search for an optimal exercise boundary. This is a one-dimensional optimization problem. In other words, the value function is given by

$$V(x) = \sup_{b \geq \max\{\ln q, x\}} V_b(x) = \sup_{b \geq \max\{\ln q, x\}} \mathbb{E}\left[ e^{-\widetilde{r}\tau_b} \left( e^{\widetilde{X}_{\tau_b}} - q \right)^+ I_{\{\tau_b < \infty\}} \middle| \widetilde{X}_0 = x \right]. \tag{15}$$

*3.1. Characterization of the function $V(x)$*

We now show that $V(x)$ is a solution of an OIDE and derive its functional form. To this end, we first introduce the C-L equation:

$$G(\beta) = \frac{\sigma^2}{2}\beta^2 + \widetilde{\mu}\beta + \lambda p \boldsymbol{\alpha}^+ (-\beta \mathbf{I} - \mathbf{T}^+)^{-1} \mathbf{t}^+ + \lambda(1-p)\boldsymbol{\alpha}^- (\beta \mathbf{I} - \mathbf{T}^-)^{-1} \mathbf{t}^- - \lambda = \widetilde{r}. \tag{16}$$

$\mathcal{B}^+$ denotes the collection of roots to the C-L equation whose real parts are larger than or equal to 1, and $\mathcal{B}^-$ collects those roots with negative real parts. The root characteristics of the C-L equation play a central role in our problem. The following properties are useful.

1. $\{e^{-rt} S_t\}_{t \geq 0} = \left\{ e^{-\widetilde{r}t} \widetilde{S}_t \right\}_{t \geq 0}$ is a martingale implying that $1 \in \mathcal{B}^+$.

2. The function $G(\beta)$ satisfies

$$\mathbb{E}\left[e^{\beta \widetilde{X}_t}\right] = e^{G(\beta)t} \tag{17}$$

where $\beta$ belongs to some bounded interval covering $[0,1]$.

If $G'(1) \geq 0$, then as shown in Section 4 $V(x) = e^x$ and $q = c$. In such a situation, a bank would have no intention of trading the stock loan with the given loan interest rate $\gamma$ and current stock price $S_0$. An interesting case occurs when $G'(1) < 0$. Thus, we focus on this latter case for the time being.

When $G'(1) < 0$, $\gamma > r$ and the effective interest rate $\widetilde{r} = r - \gamma$ is negative, which can be proven by utilizing the definition of $G(\beta)$ in (17):

$$G(\beta) = \frac{\sigma^2}{2}\beta^2 + \left(r - \gamma - \frac{\sigma^2}{2} - \lambda\left(\mathbb{E}\left[e^{Y_1}\right] - 1\right)\right)\beta + \lambda\left(\mathbb{E}\left[e^{\beta Y_1}\right] - 1\right). \tag{18}$$

Hence,

$$G'(1) = r - \gamma + \frac{\sigma^2}{2} + \lambda \mathbb{E}\left[Y_1 e^{Y_1} - e^{Y_1} + 1\right]. \tag{19}$$

As $ye^y - e^y + 1 \geq 0$ for all $y \in \mathbb{R}$, $G'(1) < 0$ implies that

$$\gamma > r + \frac{\sigma^2}{2} + \lambda \mathbb{E}\left[Y_1 e^{Y_1} - e^{Y_1} + 1\right] \geq r.$$

As the effective interest rate $\widetilde{r}$ is negative, the Wiener-Hopf factorization techniques of Mordecki (2002) and Asmussen et al. (2004) for perpetual American options subject to Lévy processes cannot be applied to the stock loan valuation problem using a phase-type Lévy model.

Without imposing a condition on $G'(1)$, the following theorem characterizes the representation of the function $V(x)$. This new result embraces the stock loan value in a double-exponential jump diffusion model (Sun, 2010) as a special case.

**Theorem 3.2.** The value function $V_b(x)$ in (15) satisfies the OIDE,

$$\begin{cases} (\mathcal{L} - \widetilde{r})\, V_b(x) & = & 0 & x < b \\ V_b(x) & = & e^x - q & x \geq b \end{cases}, \tag{20}$$

for any given $b \in \mathbb{R}$, where

$$\mathcal{L}h(x) = \frac{\sigma^2}{2}\frac{d^2 h}{dx^2}(x) + \tilde{\mu}\frac{dh}{dx}(x) + \lambda \int_{-\infty}^{\infty} \left(h(x+y) - h(x)\right) f_Y(y)dy.$$

Furthermore, the optimal value function takes the form

$$V(x)=\begin{cases}\sum\limits_{\beta_j\in\mathcal{B}^+}\omega_j e^{\beta_j x} & x<b^* \\ e^x-q & x\ge b^*\end{cases} \tag{21}$$

for some $\omega_j,\ j\in\{i\mid\beta_i\in\mathcal{B}^+\}$ to be determined according to the model, where $b^*$ is the optimal exercise boundary obtained from (15).

*Proof.* An application of the Feymann-Kac formula to $V_b(x)$ in (15) with respect to the stock price dynamics (7) produces the OIDE of (20) for a given $b$. Consider the following function as a candidate solution to the OIDE with $b=b^*$.

$$u(x)=\begin{cases}\sum\limits_{\beta_j\in\mathcal{B}^+}\omega_j e^{\beta_j x} & x<b^* \\ e^x-q & x\ge b^*\end{cases},$$

where $\omega_j,\ j\in\{i\mid\beta_i\in\mathcal{B}^+\}$ are chosen such that $u(\cdot)$ satisfies the conditions described in Lemma 3.1. In particular, we have $(e^x-q)^+\le u(x)\le e^x$ for all $x\in\mathbb{R}$. This obviously satisfies the governing equation of the OIDE (20). However, it may not be continuously differentiable at $b^*$. To get around this problem, we construct a sequence of functions $\{u_n(x)\}_{n=1}^{\infty}$ such that the following hold true.

1. $u_n(x)$ is twice continuously differentiable for all $n\in\mathbb{N}$.
2. For $x\le b^*$ or $x\ge b^*+\frac{1}{n}$, $u_n(x)\equiv u(x)$.
3. For $b^*\le x\le b^*+\frac{1}{n}$, $0\le u_n(x)\le M_1$, where $M_1$ is a positive constant.

This sequence of functions has the limit $u(x)$ for $n\to\infty$ because of the continuity of $u(x)$ shown in Lemma 3.1.

For any $x<b^*$, we have

$$\left(\mathcal{L}-\widetilde{r}\right)u_n(x)=\lambda\int_{b^*-x}^{b^*-x+1/n}\left[u_n(x+y)-u(x+y)\right]f_Y(y)dy. \tag{22}$$

Using the fact that

$$|u_n(x)-u(x)|\le\max_{x\in(b^*,b^*+1/n]}|u_n(x)|+\max_{x\in(b^*,b^*+1/n]}|u(x)|\le M_2,$$

where $M_2=M_1+e^{b^*+1}$, we have

$$\begin{aligned}|\mathcal{L}u_n(x)-\tilde{r}u_n(x)| &\le \lambda p\boldsymbol{\alpha}^+\mathbf{t}^+\int_{b^*-x}^{b^*-x+1/n}\left[u_n(x+y)-u(x+y)\right]dy \\ &\le \frac{\lambda p\boldsymbol{\alpha}^+\mathbf{t}^+M_2}{n}\to 0 \text{ uniformly for all } x<b^*\text{ , as } n\to\infty.\end{aligned} \tag{23}$$

Applying Itō's formula to $\left\{e^{-\tilde{r}t}u_n(\tilde{X}_t)\right\}_{t\geq 0}$, we obtain a sequence of local martingales $\left\{M_t^{(n)}\right\}_{t\geq 0}$ for $n\in\mathbb{N}$ as follows.

$$M_t^{(n)} = e^{-\tilde{r}(t\wedge\tau_{b^*})}u_n(\tilde{X}_{t\wedge\tau_{b^*}}) - u_n(x) - \int_0^{t\wedge\tau_{b^*}} e^{-\tilde{r}s}\left[\left(\mathcal{L}-\tilde{r}\right)u_n(\tilde{X}_s)\right]ds. \quad (24)$$

We claim that it is a martingale for any $n\in\mathbb{N}$. For any $t\geq 0$,

$$\begin{aligned}
& |e^{-\widetilde{r}(t\wedge\tau_{b^*})}u_n(\widetilde{X}_{t\wedge\tau_{b^*}})| \\
\leq\ & |e^{-\widetilde{r}t}u_n(\widetilde{X}_t)I_{\{t<\tau_{b^*}\}}| + |e^{-\widetilde{r}t}u_n(\widetilde{X}_{\tau_{b^*}})I_{\{t\geq\tau_{b^*},\widetilde{X}_{\tau_{b^*}}<b+1/n\}}| \\
& \quad + |e^{-\widetilde{r}t}u_n(\widetilde{X}_{\tau_{b^*}})I_{\{t\geq\tau_{b^*},\widetilde{X}_{\tau_{b^*}}\geq b+1/n\}}| \\
\leq\ & |e^{-\widetilde{r}t}u(\widetilde{X}_t)I_{\{t<\tau_{b^*}\}}| + M_1e^{-\widetilde{r}t} + e^{-\widetilde{r}t}(e^{\widetilde{X}_{\tau_{b^*}}}-q)I_{\{\tau_{b^*}<\infty\}}. \qquad (25)
\end{aligned}$$

By the definition in (24) and noting (23) and (25), we establish the inequality:

$$\begin{aligned}
|M_t^{(n)}| \ \leq\ & |e^{-\widetilde{r}t}u(\widetilde{X}_t)I_{\{t<\tau_{b^*}\}}| + |u_n(x)| + M_1e^{-\widetilde{r}t} \\
& + e^{-\widetilde{r}t}(e^{\widetilde{X}_{\tau_{b^*}}}-q)I_{\{\tau_{b^*}<\infty\}} - \frac{\lambda p\boldsymbol{\alpha}^+\mathbf{t}^+M_2\left(e^{-\widetilde{r}t}-1\right)}{n\tilde{r}}. \qquad (26)
\end{aligned}$$

For the first term on the right-hand side of (26), we have, for any fixed $T>0$,

$$\begin{aligned}
& \mathbb{E}_x\left[\sup_{t\in[0,T]} e^{-\widetilde{r}t}u(\widetilde{X}_t)I_{\{t<\tau_{b^*}\}}|\right] \\
\leq\ & e^{-\widetilde{r}T}\mathbb{E}_x\left[e^{\sup_{t\in[0,T]}\widetilde{X}_t}\right] \\
\leq\ & e^{-\widetilde{r}T}\mathbb{E}_x\left[e^{x+\widetilde{\mu}T+\sigma\sup_{t\in[0,T]}W_t+\sum_{i=1}^{N_T}Y_i^+}\right] \\
=\ & 2\Phi(\sigma\sqrt{T})\exp\left(-\widetilde{r}T + x + \widetilde{\mu}T + \frac{\sigma^2T}{2} + p\lambda T\boldsymbol{\alpha}^+\left(-\mathbf{I}-\mathbf{T}^+\right)^{-1}\mathbf{t}^+\right) \\
<\ & \infty,
\end{aligned}$$

where $\Phi(\cdot)$ is the cumulative distribution function of the standard normal distribution. It is now easy to see that

$$\mathbb{E}_x\sup_{t\in[0,T]}|M_t^{(n)}| < \infty, \qquad (27)$$

which guarantees that $M_t^{(n)}$ is a true martingale for all $n$. For $x < b^*$,

$$\begin{aligned} u(x) &= \lim_{n\to\infty} u_n(x) \\ &= \lim_{n\to\infty} \mathbb{E}_x \left[ e^{-\widetilde{r}(t\wedge\tau_{b^*})} u_n(\widetilde{X}_{t\wedge\tau_{b^*}}) \right] - \lim_{n\to\infty} \mathbb{E}_x M_t^{(n)} \\ &\quad - \lim_{n\to\infty} \mathbb{E}_x \left[ \int_0^{t\wedge\tau_{b^*}} e^{-\widetilde{r}s} \left[ (\mathcal{L} - \widetilde{r})\, u_n(\widetilde{X}_s) \right] ds \right] \\ &= \mathbb{E}_x \left[ e^{-\widetilde{r}(t\wedge\tau_{b^*})} u(\widetilde{X}_{t\wedge\tau_{b^*}}) \right], \end{aligned}$$

where the last equality is a consequence of the dominated convergence theorem (DCT). Let $t \to \infty$ and apply Fatou's lemma to obtain

$$\begin{aligned} u(x) &= \lim_{t\to\infty} \mathbb{E}_x \left[ e^{-\widetilde{r}(t\wedge\tau_{b^*})} u(\widetilde{X}_{t\wedge\tau_{b^*}}) \right] \\ &= \lim_{t\to\infty} \mathbb{E}_x \left[ e^{-\widetilde{r}(t\wedge\tau_{b^*})} u(\widetilde{X}_{t\wedge\tau_{b^*}}) I_{\{\tau_{b^*}<\infty\}} \right] \\ &\quad + \lim_{t\to\infty} \mathbb{E}_x \left[ e^{-\widetilde{r}(t\wedge\tau_{b^*})} u(\widetilde{X}_{t\wedge\tau_{b^*}}) I_{\{\tau_{b^*}=\infty\}} \right] \\ &\geq \mathbb{E}_x \left[ e^{-\widetilde{r}(\tau_{b^*})} u(\widetilde{X}_{\tau_{b^*}}) I_{\{\tau_{b^*}<\infty\}} \right] = \mathbb{E}_x \left[ e^{-\widetilde{r}(\tau_{b^*})} (e^{\widetilde{X}_{\tau_{b^*}}} - q) I_{\{\tau_{b^*}<\infty\}} \right]. \end{aligned}$$

In addition,

$$\begin{aligned} \mathbb{E}_x \left[ e^{-\widetilde{r}(t\wedge\tau_{b^*})} u(\widetilde{X}_{t\wedge\tau_{b^*}}) \right] &= \mathbb{E}_x \left[ e^{-\widetilde{r}(t\wedge\tau_{b^*})} (e^{\widetilde{X}_{t\wedge\tau_{b^*}}} - q) I_{\{\tau_{b^*}\leq t\}} \right] \\ &\quad + \mathbb{E}_x \left[ e^{-\widetilde{r}(t\wedge\tau_{b^*})} u(\widetilde{X}_{t\wedge\tau_{b^*}}) I_{\{\tau_{b^*}>t\}} \right]. \end{aligned} \tag{28}$$

As

$$e^{-\widetilde{r}(t\wedge\tau_{b^*})} (e^{\widetilde{X}_{t\wedge\tau_{b^*}}} - q) I_{\{\tau_{b^*}\leq t\}} \leq e^{-\widetilde{r}(t\wedge\tau_{b^*})} (e^{\widetilde{X}_{t\wedge\tau_{b^*}}} - q) I_{\{\tau_{b^*}<\infty\}}$$

and

$$\mathbb{E}_x \left[ e^{-\widetilde{r}(t\wedge\tau_{b^*})} (e^{\widetilde{X}_{t\wedge\tau_{b^*}}} - q) I_{\{\tau_{b^*}<\infty\}} \right] < \infty,$$

the DCT implies that the first term on the right-hand side of (28) converges to

$$\mathbb{E}_x \left[ e^{-\widetilde{r}(\tau_{b^*})} (e^{\widetilde{X}_{\tau_{b^*}}} - q) I_{\{\tau_{b^*}<\infty\}} \right]$$

when $t \to \infty$.

For the second term, we claim that

$$\mathbb{E}_x \left[ e^{-\widetilde{r}(t\wedge\tau_{b^*})} u(\widetilde{X}_{t\wedge\tau_{b^*}}) I_{\{\tau_{b^*}>t\}} \right] \to 0,$$

as $t \to \infty$. Consider the following two cases.

*Case 1:* $G'(1) < 0$. There exists a $\kappa_0 > 1$ such that $G(\kappa_0) - \widetilde{r} < 0$. In addition, there exists a $C_0 > 0$ such that $u(x) < C_0 e^{\kappa_0 x}$ for all $x < b^*$. Hence,

$$\begin{aligned}
\mathbb{E}_x\left[e^{-\widetilde{r}(t\wedge\tau_{b^*})}u(\widetilde{X}_{t\wedge\tau_{b^*}})I_{\{\tau_{b^*}>t\}}\right] &< \mathbb{E}_x\left[C_0 e^{-\widetilde{r}t+\kappa_0\widetilde{X}_t}I_{\{\tau_{b^*}>t\}}\right] \\
&\leq \mathbb{E}_x\left[C_0 e^{-\widetilde{r}t+\kappa_0\widetilde{X}_t}\right] \\
&\to 0 \text{ when } t \to \infty.
\end{aligned}$$

*Case 2:* $G'(1) \geq 0$. As $V(x) \leq e^x$ for all $x \in \mathbb{R}$, we have

$$\mathbb{E}_x\left[e^{-\widetilde{r}(t\wedge\tau_{b^*})}u(\widetilde{X}_{t\wedge\tau_{b^*}})I_{\{\tau_{b^*}>t\}}\right] \leq \mathbb{E}_x\left[e^{-\widetilde{r}t+\widetilde{X}_t}I_{\{\tau_{b^*}>t\}}\right]. \tag{29}$$

Consider the following probability measure $\widehat{\mathbb{P}}$

$$\frac{d\widehat{\mathbb{P}}}{d\mathbb{P}} = e^{-\widetilde{r}t+\widetilde{X}_t}. \tag{30}$$

In Appendix A of Asmussen et al. (2004), it is shown that $\{\widetilde{X}_t\}_{t>0}$ still follows a phase-type Lévy process under $\widehat{\mathbb{P}}$, but its Lévy exponent is replaced by

$$\widehat{G}(s) = G(1+s) - G(1). \tag{31}$$

Therefore,

$$\mathbb{E}_x\left[e^{-\widetilde{r}t+\widetilde{X}_t}I_{\{\tau_{b^*}>t\}}\right] = \widehat{\mathbb{E}}_x\left[I_{\{\tau_{b^*}>t\}}\right] \to \widehat{\mathbb{P}}_x\left(\tau_{b^*} = \infty\right), \tag{32}$$

as $t \to \infty$. However, the fact that $\widehat{G}'(0) = G'(1) \geq 0$ and $\widehat{G}(0) = 0$ implies that

$$\widehat{\mathbb{P}}_x(\tau_{b^*} < \infty) = \lim_{\widetilde{r}\to 0}\widehat{\mathbb{E}}_x\left[e^{-\widetilde{r}\tau_{b^*}}\right] = 1, \tag{33}$$

which proves our claim. Therefore, the candidate solution $u(\cdot)$ is indeed the solution. $\square$

Theorem 3.2 can be further streamlined to express the solution in the most important situation, that is, $G'(1) < 1$. Inspired by Zhang and Zhou (2009), we now show that the solution of $V(x)$ should exclude the term corresponding to $\beta_j = 1$ in Theorem 3.2 under such a condition.

**Proposition 3.1.** If $G'(1) < 0$, then the value function in Theorem 3.2 becomes

$$V(x) = \begin{cases} \sum_{\beta_j \in \mathcal{B}^+ \setminus \{1\}} \omega_j e^{\beta_j x} & x < b^* \\ e^x - q & x \geq b^* \end{cases}.$$

*Proof.* Denote $j_0$ as the index such that $\beta_{j_0} = 1$. Then, we need to show that $\omega_{j_0} = 0$. For $t \leq \tau_{b^*}$,

$$\mathbb{E}\left[e^{-\widetilde{r}t} V(\widetilde{X}_t) \mid \widetilde{X}_0 = x\right] = V(x) + \int_0^t e^{-\widetilde{r}s} \left(\mathcal{L} - \widetilde{r}\right) V(\widetilde{X}_s) ds = V(x).$$

Hence, for any $T > 0$,

$$\begin{aligned} V(x) &= \mathbb{E}\left[e^{-\widetilde{r}\tau_{b^*} \wedge T} V(\widetilde{X}_{\tau_{b^*} \wedge T}) \mid \widetilde{X}_0 = x\right] \\ &\leq \mathbb{E}\left[e^{-\widetilde{r}\tau_{b^*}} V(\widetilde{X}_{\tau_{b^*}}) I_{\{\tau_{b^*} < T\}} \mid \widetilde{X}_0 = x\right] + \mathbb{E}\left[e^{-\widetilde{r}T} V(\widetilde{X}_T) I_{\{\tau_{b^*} \geq T\}} \mid \widetilde{X}_0 = x\right]. \end{aligned}$$

It is clear that the first term converges to

$$\mathbb{E}\left[e^{-\widetilde{r}\tau_{b^*}} \left(e^{\widetilde{X}_{\tau_{b^*}}} - q\right) I_{\{\tau_{b^*} < \infty\}} \mid \widetilde{X}_0 = x\right]$$

as $T \to \infty$. It remains to show that the second term also converges to zero as $T \to \infty$.

By Theorem 3.2, $V(x)$ is a linear combination of $e^{\beta_i x}$ for $x < b^*$. Consider the validity of

$$\mathbb{E}\left(e^{-\widetilde{r}T} e^{\kappa \widetilde{X}_T}\right) \to 0 \text{ as } T \to \infty$$

for different values of $\kappa$.

Note that $\mathbb{E}\left(e^{-\widetilde{r}T} e^{\kappa \widetilde{X}_T}\right) = e^{(G(\kappa) - \widetilde{r})T}$. For $\kappa = 1$, the expectation becomes $e^0 = 1$ and does not converge to zero. Hence, the term $\omega_{j_0} e^x$ should be removed from the linear combination by setting its coefficient to zero.

As $G'(1) < 0$, there exists a $\kappa_0 > 1$ such that $G(\kappa_0) - \widetilde{r} < 0$. In addition, for any $\beta_i \in \mathcal{B}^+ \setminus \{1\}$, there exists a $K_i > 0$ such that $e^{\beta_i x} \leq K_i e^{\kappa_0 x}$ for $x \in (-\infty, b^*)$. Therefore, $\mathbb{E}\left(e^{-\widetilde{r}T} e^{\beta_i \widetilde{X}_T}\right) \leq K_i \mathbb{E}\left(e^{-\widetilde{r}T} e^{\kappa_0 \widetilde{X}_T}\right) \to 0$ as $T \to \infty$. $\square$

We conclude this section by summarizing its major findings and the remaining tasks to be investigated in the next section. For any given exponential phase-type Lévy model, the stock loan valuation formula has the general representation in

Theorem 3.2. If $G'(1) \geq 0$, then the stock loan will not be traded, a scenario that is addressed in the next section. If $G'(1) < 0$, then the stock loan pricing formula is given by Proposition 3.1, which involves solving the roots of the C-L equation (16), identifying the roots with a real part greater than 1, and determining the optimal exercise boundary $b^*$. However, the characterization of the real part of the roots of the C-L equation (16) is in general not straightforward. The following section presents several important special cases for which the root characteristics are fully investigated and the stock loan value is obtained as a closed-form solution.

## 4. Valuation formulas

We first solve the problem under the assumption of hyperexponential jump diffusions, a special case of phase-type jump diffusions. Although the hyperexponential jump diffusion model has been investigated by Asmussen et al. (2007) for equity default swaps, by Cai (2009) for the first passage time problem, and by Cai and Kou (2011) for barrier and lookback options, these authors consider only the case of a positive interest rate. The optimal exercise boundary of a stock loan has yet to be investigated. We extend the solution from this hyperexponential case to a fairly general class of phase-type jump diffusion models by constructing a transformation using the argument principle. Finally, we analyze the stock loan value when $G'(1) \geq 0$.

### *4.1. Hyperexponential jumps*

Suppose that $\mathbf{T}^+$ and $\mathbf{T}^-$ take the following forms

$$\mathbf{T}^+ = \begin{pmatrix} -\eta_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & -\eta_m \end{pmatrix}, \quad \mathbf{T}^- = \begin{pmatrix} -\theta_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & -\theta_n \end{pmatrix}, \tag{34}$$

where $\eta_i > 1$ for $i = 1, \ldots, m$ and $\theta_k > 0$ for $k = 1, \ldots, n$. This phase-type distribution is reduced to a hyperexponential distribution. The following proposition summarizes the root characteristics of the C-L equation (16).

**Lemma 4.1.** If $G'(1) < 0$, then the C-L equation $G(\beta) = \widetilde{r}$ in (16), in which $\mathbf{T}^+$ and $\mathbf{T}^-$ are defined as in (34), has exactly $n$ distinct negative real roots and $m+2$ distinct real roots that are no less than 1.

*Proof.* Under the hyperexponential jump diffusion model,

$$G(\beta) = \frac{\sigma^2}{2}\beta^2 + \tilde{\mu}\beta + \lambda p \sum_{i=1}^{m} \frac{\alpha_i^+ \eta_i}{\eta_i - \beta} + \lambda(1-p) \sum_{j=1}^{n} \frac{\alpha_i^- \theta_j}{\theta_j + \beta} - \lambda. \tag{35}$$

The followings properties are clear.

1. $G(0) = 0$;
2. $G(\infty) = \infty$, $G(-\infty) = \infty$;
3. $G(\eta_i-) = \infty$, $G(\eta_i+) = -\infty$ for $i = 1, \ldots, m$;
4. $G(-\theta_j-) = -\infty$, $G(-\theta_j+) = \infty$ for $j = 1, \ldots, n$; and
5. $G(\beta)$ is continuous except for the values $\eta_i$, $i = 1, \ldots, m$ and $-\theta_j$, $j = 1, \ldots, n$,

where $G(u\pm) = \lim_{x \to u\pm} G(x)$. All of these imply that $G(\beta) = \tilde{r}$ has at least one root in each of the intervals

$$(-\infty, -\theta_n), (-\theta_n, -\theta_{n-1}), \ldots, (-\theta_2, -\theta_1), (\eta_1, \eta_2), \ldots, (\eta_{m-1}, \eta_m), (\eta_m, \infty).$$

Moreover, $G(\beta) = \tilde{r}$ has the same number of roots as the $m + n + 2$ degree polynomial

$$(G(\beta) - \tilde{r}) \prod_{i=1}^{m} (\eta_i - \beta) \prod_{j=1}^{n} (\theta_j + \beta).$$

Therefore, it has at most $m + n + 2$ real roots.

As $G(\beta)$ is decreasing over the interval $(-\theta_1, 0)$, $G(0) = 0$ and $G(-\theta_1+) = \infty$, there is no root in the interval $(-\theta_1, 0)$. As 1 is always a root, and complex roots always exist in pairs, there are two real roots in the interval $(0, \eta_1)$. The assumption that $G'(1) < 0$ ensures that the two real roots in $(0, \eta_1)$ are distinct and that both are no less than 1. □

By Proposition 3.1, the value function of a stock loan takes the form, $V(x) = \sup_{b \geq \max(\ln q, x)} V_b(x)$, where

$$V_b(x) = \begin{cases} \sum\limits_{j=1}^{m+1} \omega_j e^{\beta_j x} & x < b \\ e^x - q & x \geq b \end{cases}, \tag{36}$$

for any given constant $b \in \mathbb{R}$, and $1 < \beta_1 < \beta_2 < \ldots < \beta_{m+1}$ are roots of the C-L equation (16). Moreover, $(\mathcal{L} - \widetilde{r})\, V_b(x) = 0$ for $x < b$. Therefore,

$$
\begin{aligned}
0 &= \mathcal{L}V_b(x) - \widetilde{r}V_b(x) \\
&= \frac{\sigma^2}{2}\frac{d^2V_b}{dx^2}(x) + \widetilde{\mu}\frac{dV_b}{dx}(x) + \lambda \int_{-\infty}^{\infty} \left(V_b(x+y) - V_b(x)\right) f_Y(y)dy - \widetilde{r}V_b(x) \\
&= \sum_{j=1}^{m+1} \omega_j e^{\beta_j x} \left(G(\beta_j) - \tilde{r}\right) - \lambda \int_{b-x}^{\infty} \sum_{j=1}^{m+1} \omega_j e^{\beta_j (x+y)} f_Y(y)dy \\
&\quad + \lambda \int_{b-x}^{\infty} \left(e^{x+y} - q\right) f_Y(y)dy \\
&= -\lambda \sum_{j=1}^{m+1} \omega_j e^{\beta_j x} \sum_{i=1}^{m} p\alpha_i^{+} \frac{\eta_i}{\eta_i - \beta_j} e^{-(\eta_i - \beta_j)(b-x)} + \lambda e^{x} \sum_{i=1}^{m} p\alpha_i^{+} \frac{\eta_i}{\eta_i - 1} e^{-(\eta_i - 1)(b-x)} \\
&\quad - \lambda q \sum_{i=1}^{m} p\alpha_i e^{-\eta_i (b-x)} \\
&= \lambda \sum_{i=1}^{m} p\alpha_i^{+} e^{\eta_i x} \left( \frac{\eta_i}{\eta_i - 1} e^{-(\eta_i - 1)b} - q e^{-\eta_i b} - \sum_{j=1}^{m+1} \omega_j \frac{\eta_i}{\eta_i - \beta_j} e^{-(\eta_i - \beta_j)b} \right).
\end{aligned} \tag{37}
$$

This implies the following $m$ linear equations for $\omega_i$.

$$
\sum_{j=1}^{m+1} \omega_j \frac{\eta_i}{\eta_i - \beta_j} e^{-(\eta_i - \beta_j)b} = \frac{\eta_i}{\eta_i - 1} e^{-(\eta_i - 1)b} - q e^{-\eta_i b}, \tag{38}
$$

for $i = 1, \ldots, m$. The continuity of $V_b(\cdot)$ at $b$ gives

$$
\sum_{j=1}^{m+1} \omega_j e^{\beta_j b} = e^b - q. \tag{39}
$$

The solution of these $m+1$ equations can be summarized as follows.

**Lemma 4.2.** The solution of the system of $m+1$ linear equations

$$
\left\{
\begin{array}{rcll}
\sum_{j=1}^{m+1} \omega_j \frac{\eta_i}{\eta_i - \beta_j} e^{-(\eta_i - \beta_j)b} &=& \frac{\eta_i}{\eta_i - 1} e^{-(\eta_i - 1)b} - q e^{-\eta_i b} & \text{for } i = 1, \ldots, m \\
\sum_{j=1}^{m+1} \omega_j e^{\beta_j b} &=& e^b - q &
\end{array}
\right. , \tag{40}
$$

is given by

$$\omega_j = \frac{\sum_{i=1}^{m+1}\left(R_i \prod_{k=1}^{m+1}(\eta_i - \beta_k) \prod_{l=1, l\neq i}^{m+1} \frac{\beta_j - \eta_l}{\eta_i - \eta_l}\right)}{\beta_j e^{\beta_j b} \prod_{k=1, k\neq j}^{m+1} (\beta_j - \beta_k)}, \quad j = 1, 2, \ldots, m+1, \tag{41}$$

where $R_i = \frac{e^b}{\eta_i - 1}$ for $i = 1, \ldots, m$ and $R_{m+1} = q - e^b$.

*Proof.* Take $\widetilde{\omega}_j = \omega_j e^{\beta_j b}$ for $j = 1, \ldots, m+1$, $\eta_{m+1} = 0$, $R_i = \frac{e^b}{\eta_i - 1}$ for $i = 1, \ldots, m$ and $R_{m+1} = q - e^b$. The linear system becomes

$$\sum_{j=1}^{m+1} \widetilde{\omega}_j \frac{\beta_j}{\eta_i - \beta_j} = R_i \ \text{ for } i = 1, \ldots, m+1. \tag{42}$$

Using a partial fraction technique similar to that in Chen et al. (2007), we have

$$\sum_{j=1}^{m+1} \frac{D_j \beta_j}{x - \beta_j} = \sum_{i=1}^{m+1} R_i \prod_{k=1}^{m+1} \frac{\eta_i - \beta_k}{x - \beta_k} \prod_{l=1, l\neq i}^{m+1} \frac{x - \eta_l}{\eta_i - \eta_l}, \tag{43}$$

where $D_j$, $j = 1, \ldots, m+1$ are the partial fraction coefficients. Multiplying (43) by $(x - \beta_k)$ on both sides and setting $x = \beta_k$ yields

$$D_k \beta_k = \frac{\sum_{i=1}^{m+1}\left(R_i \prod_{j=1}^{m+1}(\eta_i - \beta_j) \prod_{l=1, l\neq i}^{m+1} \frac{\beta_k - \eta_l}{\eta_i - \eta_l}\right)}{\prod_{j=1, j\neq k}^{m+1} (\beta_k - \beta_j)}. \tag{44}$$

When $x = \eta_i$ in (43), we have

$$\sum_{j=1}^{m+1} \frac{D_j \beta_j}{\eta_i - \beta_j} = R_i \prod_{k=1}^{m+1} \frac{\eta_i - \beta_k}{\eta_i - \beta_k} \prod_{l=1, l\neq i}^{m+1} \frac{\eta_i - \eta_l}{\eta_i - \eta_l} = R_i.$$

Hence, $\widetilde{\omega}_j = D_j$, $j = 1, \ldots, m+1$, and the result follows. □

After obtaining the coefficients in (36), the remaining task is to determine the optimal exercise boundary $b^*$ that maximizes the candidate solution $V_b(x)$. The following identity is useful for that purpose.

**Lemma 4.3.** If $\{\beta_k\}_{k=1}^{m+1}$ and $\{\eta_i\}_{i=1}^{m+1}$ are all distinct, then

$$\prod_{k=1,k\neq j}^{m+1}(\beta_k-1)=\sum_{k=1}^{m+1}\prod_{k=1,k\neq j}^{m+1}(\beta_k-\eta_i)\prod_{l=1,l\neq i}^{m+1}\left(\frac{\eta_l-1}{\eta_l-\eta_i}\right). \tag{45}$$

*Proof.* Consider the following polynomial.

$$P_j(x)=\prod_{k=1,k\neq j}^{m+1}(\beta_k-1-x)\quad\text{for } j=1,\ldots,m+1, \tag{46}$$

which are of degree $m$. It is clear that

$$P_j(\eta_i-1)=\prod_{k=1,k\neq j}^{m+1}(\beta_k-\eta_i)\,. \tag{47}$$

By Lagrange interpolation,

$$\begin{aligned} L_j(x) &= \sum_{i=1}^{m+1}P_j(\eta_i-1)\prod_{l=1,l\neq i}^{m+1}\left(\frac{\eta_l-1-x}{\eta_l-\eta_i}\right) \\ &= \sum_{i=1}^{m+1}\prod_{k=1,k\neq j}^{m+1}(\beta_k-\eta_i)\prod_{l=1,l\neq i}^{m+1}\left(\frac{\eta_l-1-x}{\eta_l-\eta_i}\right) \end{aligned} \tag{48}$$

is a polynomial of degree $m$ passing through all points in the set

$$\{(\eta_i-1,P_j(\eta_i-1))\}_{i=1}^{m+1}\,.$$

As $P_j(x)$ is a polynomial of degree $m$, and it matches the value of $L_j(x)$ at $m+1$ points, we have

$$P_j(x)=L_j(x)\quad\forall x\in\mathbb{R}.$$

The result follows by setting $x=0$. □

**Proposition 4.1.** Consider the stock loan valuation problem (10) with the underlying stock price process of (7) and (8), in which $\mathbf{T}^+$ and $\mathbf{T}^-$ are defined as in (34). If $G'(1)<0$, then the stock loan pricing formula is given by (36) in which $1<\beta_1<\ldots<\beta_{m+1}$ are distinct roots of the C-L equation (16), the coefficients $\{\omega_1,\ldots,\omega_{m+1}\}$ are obtained from Lemma 4.2, and the optimal exercise boundary is given by

$$b^*=\ln\left(q\prod_{k=1}^{m+1}\frac{\beta_k}{\beta_k-1}\prod_{l=1}^{m}\frac{\eta_l-1}{\eta_l}\right). \tag{49}$$

*Proof.* By Proposition 3.1, the stock loan pricing formula takes the form of (36) once $\beta_j$, $j = 1, \ldots, m+1$, are distinct roots of the C-L equation (16) over the interval $(1, \infty)$, which is confirmed by Lemma 4.1. Lemma 4.2 then provides explicit expressions for the coefficients $\omega_j$, $j = 1, \ldots, m+1$, in (36).

The remaining task is to determine the $b^*$ that maximizes the value function $V_b(x)$ in (36). It suffices for us to maximize the function on the interval $(-\infty, x)$. Consider

$$\frac{d}{db}V_b(x) = \sum_{j=1}^{m+1} e^{\beta_j(x-b)} \left(\frac{d}{db}\widetilde{\omega_j} - \widetilde{\omega_j}\beta_j\right),$$

where $\widetilde{\omega_j} = \omega_j e^{\beta_j b}$. Simple algebra shows that

$$\begin{aligned}
\frac{d}{db}\widetilde{\omega_j} - \widetilde{\omega_j}\beta_j \;&=\; \frac{(1-\beta_j)\, e^b \left[\sum_{j=1}^{m} \frac{1}{\eta_i - 1} \prod_{k=1}^{m+1} (\eta_i - \beta_k) \prod_{l=1, l\neq i}^{m+1} \frac{\beta_j - \eta_l}{\eta_i - \eta_l} + \prod_{k=1}^{m+1} \beta_k \prod_{l=1}^{m} \frac{\beta_j - \eta_l}{\eta_l}\right]}{\beta_j \prod_{k=1, k\neq j}^{m+1} (\beta_j - \beta_k)} \\
&\quad + \frac{q \prod_{k=1}^{m+1} \beta_k \prod_{l=1}^{m} \frac{\beta_j - \eta_l}{\eta_l}}{\prod_{k=1, k\neq j}^{m+1} (\beta_j - \beta_k)}.
\end{aligned}$$

Hence, $\frac{d}{db}\widetilde{\omega_j} - \widetilde{\omega_j}\beta_j = 0$ if and only if

$$\begin{aligned}
\frac{e^b}{q} \;&=\; \frac{\beta_j}{\beta_j - 1} \frac{\prod_{k=1}^{m+1} \beta_k \prod_{l=1}^{m} \frac{\beta_j - \eta_l}{\eta_l}}{\sum_{j=1}^{m} \frac{1}{\eta_i - 1} \prod_{k=1}^{m+1} (\eta_i - \beta_k) \prod_{l=1, l\neq i}^{m+1} \frac{\beta_j - \eta_l}{\eta_i - \eta_l} + \prod_{k=1}^{m+1} \beta_k \prod_{l=1}^{m} \frac{\beta_j - \eta_l}{\eta_l}} \\
&=\; \frac{1}{\beta_j - 1} \frac{\prod_{k=1}^{m+1} \beta_k \prod_{l=1}^{m} \frac{\eta_l - 1}{\eta_l}}{\sum_{j=1}^{m+1} \prod_{k=1, k\neq j}^{m+1} (\beta_k - \eta_i) \prod_{l=1, l\neq i}^{m+1} \left(\frac{\eta_l - 1}{\eta_l - \eta_i}\right)} \\
&=\; \prod_{k=1}^{m+1} \frac{\beta_k}{\beta_k - 1} \prod_{l=1}^{m} \frac{\eta_l - 1}{\eta_l},
\end{aligned}$$

where the last equality is an application of Lemma 4.3. Hence, the result follows. □

*4.2. Phase-type jumps*

We are now ready to extend the previous results to the case of phase-type Lévy models. Suppose that $\mathbf{T}^+$ and $\mathbf{T}^-$ are symmetric (and hence diagonalizable) matrices that have distinct eigenvalues. Thus, there exist orthogonal matrices $\mathbf{Q}^+$ and $\mathbf{Q}^-$ such that

$$\mathbf{T}^+ = \left(\mathbf{Q}^+\right)^T \mathbf{\Lambda}^+\mathbf{Q}^+ \quad \text{and} \quad \mathbf{T}^- = \left(\mathbf{Q}^-\right)^T \mathbf{\Lambda}^-\mathbf{Q}^- \ , \tag{50}$$

where

$$\mathbf{\Lambda}^+ = \begin{pmatrix} -\eta_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & -\eta_m \end{pmatrix}, \quad \mathbf{\Lambda}^- = \begin{pmatrix} -\theta_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & -\theta_n \end{pmatrix} .$$

The following characterizes the roots of the C-L equation (16) corresponding to the stock price process of (7) and (8) using the matrices in (50).

**Theorem 4.1.** Suppose that the Lévy exponent $G(\beta)$ in (16) uses $\mathbf{T}^+$ and $\mathbf{T}^-$ in (50). If $G'(1) < 0$, then the C-L equation $G(\beta) = \widetilde{r}$ has exactly $m+1$ roots in the complex domain $\mathcal{D}_+ = \{z \in \mathbb{C} | Re(z) > 1\}$ and exactly $n$ roots in the complex domain $\mathcal{D}_- = \left\{z \in \mathbb{C} | Re(z) < \max_i \{-\theta_i\}\right\}$.

*Proof.* Consider the following functions.

$$\begin{aligned} \text{Let} \quad f_0(z) &= \tilde{\mu}z + \tfrac{\sigma^2}{2}z^2 + \lambda p \left(\boldsymbol{\alpha}^+ \left(-z\mathbf{I} - \mathbf{\Lambda}^+\right)^{-1} (-\mathbf{\Lambda}^+)\mathbf{1} - 1\right) \\ &\quad + \lambda(1-p) \left(\boldsymbol{\alpha}^- \left(-z\mathbf{I} - \mathbf{\Lambda}^-\right)^{-1} (-\mathbf{\Lambda}^-)\mathbf{1} - 1\right) - \tilde{r}, \\ f_1(z) &= \tilde{\mu}z + \tfrac{\sigma^2}{2}z^2 + \lambda p \left(\boldsymbol{\alpha}^+ \left(\mathbf{Q}^+\right)^T \left(-z\mathbf{I} - \mathbf{\Lambda}^+\right)^{-1} (-\mathbf{\Lambda}^+)\mathbf{Q}^+\mathbf{1} - 1\right) \\ &\quad + \lambda(1-p) \left(\boldsymbol{\alpha}^- \left(\mathbf{Q}^-\right)^T \left(-z\mathbf{I} - \mathbf{\Lambda}^-\right)^{-1} (-\mathbf{\Lambda}^-)\mathbf{Q}^-\mathbf{1} - 1\right) - \tilde{r}, \\ f_t(z) &= \left[f_0(z)\right]^{(1-t)} \left[f_1(z)\right]^t \qquad \text{for} \quad t \in (0,1). \end{aligned}$$

Thus, $f_t(z)$ has $m$ poles $\eta_1, \ldots, \eta_m$ in $\mathcal{D}_+$ for all $t \in [0,1]$. From the hyperexponential case (Proposition 4.1), we know that $f_0(t)$ has $m+1$ zeros in $\mathcal{D}_+$.

We construct a boundary strip $\mathcal{C}_+$ of $\mathcal{D}_+$ such that $f_t(z)$ has no zero on it. As $|f_t(z)| \to \infty$ as $|z| \to \infty$ for $t = 0, 1$, there exists $R \in \mathbb{R}$ such that all roots of $f_t(z) = 0,\ t \in (0,1)$ are in the region

$$\mathcal{D}_R = \{z \in \mathbb{C} : Re(z) \geq 0,\ |z| \leq R\} . \tag{51}$$

Alternatively, as $G'(1) < 0$, there exists $\kappa_1 \in \mathbb{R},\ \kappa_1 > 1$ (in fact, we can set $\kappa_1$ arbitrarily close to 1), such that $f_t(\kappa_1) = Re(f_t(\kappa_1)) < 0$. For $t = 0, 1,\ \nu \in \mathbb{R}$,

$$\begin{aligned} e^{Re(f_t(\kappa_1+i\nu))} &= \left|e^{f_t(\kappa_1+i\nu)}\right| \\ &= \left|\mathbb{E}\left(e^{(\kappa_1+i\nu)X_1-\widetilde{r}}\right)\right| \\ &\le \mathbb{E}\left(e^{(\kappa_1)X_1-\widetilde{r}}\right) \\ &= e^{Re(f_t(\kappa_1))} \\ &< 1. \end{aligned}$$

Hence, we have $Re(f_t(\kappa_1 + i\nu)) < 0\ \ \forall \nu \in \mathbb{R}$, which gives the boundary strip

$$\mathcal{C}_+ = \{z \in \mathbb{C} : |z| = R,\ Re(z) \ge \kappa_1\} \cup \{z \in \mathbb{C} : Re(z) = \kappa_1,\ -R \le Im(z) \le R\}\,.$$

By the continuity of $f_t(z)$ and the argument principle, we deduce that

$$n_t = \frac{1}{2\pi i}\oint_{\mathcal{C}_+} \frac{f_t'(z)}{f_t(z)}dz \tag{52}$$

is integer-valued and continuous over $t \in [0, 1]$. Hence, $n_0 = n_1$, i.e., $f_1(z)$ has $m + 1$ zeros in $\mathcal{D}_+$. This completes the proof of the first part of the statement.

To show the second part of the statement, we repeat the foregoing arguments with the boundary strip,

$$\mathcal{C}_- = \{z \in \mathbb{C} : |z| = R,\ Re(z) \le \kappa_2\} \cup \{z \in \mathbb{C} : Re(z) = \kappa_2,\ -R \le Im(z) \le R\}\,,$$

where $\kappa_2 \in \mathbb{R}$ and $\kappa_2 < \max_i\{-\theta_i\}$ is chosen arbitrarily close to $\max_i\{-\theta_i\}$. $\square$

**Proposition 4.2.** Consider the stock loan valuation problem (10) with the underlying stock price process of (7) and (8), in which $\mathbf{T}^+$ and $\mathbf{T}^-$ are defined as in (50). If $G'(1) < 0$ and the $m + 1$ roots of the C-L equation (16) in $\mathcal{D}_+$ are all distinct, then the stock loan pricing formula is given by

$$V(x) = \begin{cases} \sum\limits_{j=1}^{m+1} \omega_j e^{\beta_j x} & x < b^* \\ e^x - q & x \ge b^* \end{cases}, \tag{53}$$

where $\{\beta_1, \ldots, \beta_{m+1}\}$ are the $m + 1$ roots in $\mathcal{D}_+$ as shown by Theorem 4.1, $\{\omega_1, \ldots, \omega_{m+1}\}$ are given by

$$\omega_j = \frac{\sum\limits_{i=1}^{m+1}\left(R_i \prod\limits_{k=1}^{m+1}(\eta_i - \beta_k) \prod\limits_{l=1, l\neq i}^{m+1} \frac{\beta_j - \eta_l}{\eta_i - \eta_l}\right)}{\beta_j e^{\beta_j b} \prod\limits_{k=1, k\neq j}^{m+1} (\beta_j - \beta_k)}, \tag{54}$$

and the optimal exercise boundary is

$$b^* = \ln\left(q\prod_{k=1}^{m+1}\frac{\beta_k}{\beta_k-1}\prod_{l=1}^{m}\frac{\eta_l-1}{\eta_l}\right). \tag{55}$$

*Proof.* By Proposition 3.1, for any given $b \in \mathbb{R}$, if there are no multiple roots in $\mathcal{D}_+$, then the value function is of the form

$$V_b(x) = \begin{cases} \sum\limits_{j=1}^{m+1}\omega_j e^{\beta_j x} & x < b \\ e^x - q & x \geq b \end{cases}. \tag{56}$$

Using this solution form, we compute the following.

$$\begin{aligned} \mathcal{L}V_b(x) - \widetilde{r}V_b(x) &= \lambda p\boldsymbol{\alpha}^+\Big[\left(\mathbf{Q}^+\right)^T e^{\boldsymbol{\Lambda}^+(b-x)}\Big(\left(-\boldsymbol{\Lambda}^+-\mathbf{I}\right)^{-1}e^{b\mathbf{I}} + q{\boldsymbol{\Lambda}^+}^{-1} \\ &\quad - \sum\nolimits_{j=1}^{m+1}\omega_j\left(-\boldsymbol{\Lambda}^+-\beta_j\mathbf{I}\right)^{-1}e^{\beta_j b\mathbf{I}}\Big)\left(-\boldsymbol{\Lambda}^+\right)\mathbf{Q}^+\Big]\mathbf{1}, \end{aligned}$$

which is equal to $0$ for $x \in (-\infty, b)$. Hence, we obtain a system of linear equations,

$$\begin{cases} \sum\limits_{j=1}^{m+1}\omega_j\frac{\eta_i}{\eta_i-\beta_j}e^{-(\eta_i-\beta_j)b} = \frac{\eta_i}{\eta_i-1}e^{-(\eta_i-1)b} - qe^{-\eta_i b} \qquad \text{for } i = 1,\dots,m \\ \sum\limits_{j=1}^{m+1}\omega_j e^{\beta_j b} = e^b - q \end{cases},$$

that takes exactly the same form as the linear system (40) in the hyperexponential case. Therefore, the coefficients are those given in (54), and the optimal exercise boundary is given by (55). □

### 4.3. *The case of $G'(1) \geq 0$*

The foregoing analysis is based on the assumption that $G'(1) < 0$. We now investigate the stock loan problem when $G'(1) \geq 0$. We first consider the hyperexponential jump diffusion model.

**Proposition 4.3.** Consider the stock price process of (7) and (8), which uses $\mathbf{T}^+$ and $\mathbf{T}^-$ as defined in (34). If $G'(1) \geq 0$, then $V(x) = e^x$.

*Proof.* Using a similar technique to the proof of Lemma 4.1, the C-L equation $G(\beta) = \widetilde{r}$ has exactly $n$ distinct negative roots, two positive roots in the interval $(0, 1]$ (at least one of which is $1$) and $m$ distinct positive roots in the interval $(\eta_1, \infty)$.

When $G'(1) = 0$, there is a double root at $1$. Thus, we use the stock loan formula with $G'(1) < 0$ and let $\beta_1$ go to $1$. This implies that $b^* \to \infty$, $\omega_j \to 0$ for $j \neq 1$ and

$$\omega_1 \to \frac{\sum_{k=1}^{m+1} \prod_{k=2}^{m+1} (\beta_k - \eta_i) \prod_{l=1, l\neq i}^{m+1} \left(\frac{\eta_l - 1}{\eta_l - \eta_i}\right)}{\prod_{k=2}^{m+1} (\beta_k - 1)} = 1,$$

where the last equality is a result of Lemma 4.3 when $j = 1$. Hence, $V(x) = e^x$.

Consider the case of $G'(1) > 0$. As the term $\omega_j e^{\beta_j x}$ for $0 < \beta_j < 1$ is rejected by convexity, the solution can be written as

$$V(x) = \begin{cases} \sum_{j=1}^{m+1} \omega_j e^{\beta_j x} & x < b \\ e^x - q & x \geq b \end{cases},$$

where $\beta_1 = 1 < \beta_2 < \ldots < \beta_{m+1}$. By Theorem 3.2, this value function satisfies the OIDE specified in that theorem. Substituting the solution form into the OIDE yields a system of linear equations the same as that in (40). The solution to that liner system is $\omega_1 = 1$, and $\omega_j = 0$ for $j = 2, \ldots, m + 1$, which implies $b^* = \infty$. Hence, $V(x) = e^x$. □

This result for hyperexponential jump diffusion can be extended to cover the case of phase-type Lévy models with $\mathbf{T}^+$ and $\mathbf{T}^-$ as defined in (50).

**Proposition 4.4.** Consider the stock price process of (7) and (8), which uses $\mathbf{T}^+$ and $\mathbf{T}^-$ as defined in (50). If $G'(1) \geq 0$ and there are $m + 1$ distinct roots in $\mathcal{D}_+$, then $V(x) = e^x$.

*Proof.* By repeating the argument of Theorem 4.1, it is easy to prove the following. If $G'(1) \geq 0$, then the C-L equation $G(\beta) = \widetilde{r}$ in (16) has exactly $n$ roots in the complex domain $\mathcal{D}_- = \left\{ z \in \mathbb{C} | Re(z) < \max_i \{-\theta_i\} \right\}$, and the number of roots in the domain $\mathcal{D}_+ = \{z \in \mathbb{C} | Re(z) \geq 1\}$ is given by

$$\begin{cases} m + 2 & \text{when } G'(1) = 0 \\ m + 1 & \text{when } G'(1) > 0 \end{cases}.$$

The result follows from using Proposition 4.3 because $b^*$, which takes the same form as that in Proposition 4.3, tends to infinity. $\square$

We conclude this section by summarizing the procedures of computing the stock loan value. For any stock price process in the form of (7) and (8) in which $\mathbf{T}^+$ and $\mathbf{T}^-$ are symmetric, we diagonalize $\mathbf{T}^+$ and $\mathbf{T}^-$ in the form of (50) and constitute the C-L equation (16). If $G'(1) \geq 0$, then the stock loan value is given by $V(x) = e^x$. Otherwise, we compute the roots of the C-L equation using the root finding procedures of a high degree polynomial. Then the solution can be computed as in (53), (54) and (55).

## 5. Conclusion

This paper reports a full investigation into the stock loan valuation problem using a fairly general class of phase-type Lévy models. As phase-type Lévy models, even though hyperexponential jump diffusion models alone, can approximate arbitrary Lévy models arbitrarily close as shown by Asmussen et al. (2007), the present paper provides important insights into stock loan valuation when the underlying stock price follows a general Lévy model. We clearly clarify the situations that banks will have no intention of offering a security lending service in a phase-type Lévy economy, and the appropriate service charge if business does occur. The variational inequality approach and the transformation connecting hyperexponential jump diffusion to a fairly general class of phase-type Lévy models may have potential alternative applications related to phase-type Lévy models. An interesting extension to the present work is to adopt the stochastic interest rate model under the framework of Wong and Zhao (2011).

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