THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH3280 Introductory Probability 2023-2024 Term 2
Homework Assignment 4
Due Date: 5 April, 2024 (Friday)

I declare that the assignment here submitted is original except for source material explicitly acknowledged, the piece of work, or a part of the piece of work has not been submitted for more than one purpose (i.e. to satisfy the requirements in two different courses) without declaration, and that the submitted soft copy with details listed in the "Submission Details" is identical to the hard copy, if any, which has been submitted. I also acknowledge that I am aware of University policy and regulations on honesty in academic work, and of the disciplinary guidelines and procedures applicable to breaches of such policy and regulations, as contained on the University website https://www.cuhk.edu.hk/policy/academichonesty/

It is also understood that assignments without a properly signed declaration by the student concerned will not be graded by the course teacher.

Signature ____________________ Date ____________________

General Regulations

- All assignments will be submitted and graded on Gradescope. You can view your grades and submit regrade requests there as well. For submitting your PDF homework on Gradescope, here are a few tips.

  Where is Gradescope?

  Do the following:

  1. Go to 2023R2 Introductory Probability (MATH3280B)
  2. Choose Tools in the left-hand column
  3. Scroll down to the bottom of the page
  4. The green Gradescope icon will be there

- Late assignments will receive a grade of 0.

- Write your COMPLETE name and student ID number legibly on the cover sheet (otherwise we will not take any responsibility for your assignments). Please write your answers using a black or blue pen, NOT any other color or a pencil.

  For the declaration sheet:

  Either

Use the attached file, sign and date the statement of Academic Honesty, convert it into a PDF and submit it with your homework assignments via Gradescope.

Or

Write your name on the first page of your submitted homework, and simply write out the sentence "I have read the university regulations."

- Write your solutions on A4 white paper or use an iPad or other similar device to present your answers and submit a digital form via Gradescope. Please do not use any colored paper and make sure that your written solutions are a suitable size (easily read). Please be aware that you can only use a ball-point pen to write your answers for any exams.

- Show all work for full credit. In most cases, a correct answer with no supporting work will NOT receive full credit. What you write down and how you write it are the most important means of your answers getting good marks on this homework. Neatness and organization are also essential.

1. The probability density function of X, the lifetime of a certain type of electronic device (measured in hours), is given by

$$f(x) = \begin{cases} \frac{10}{x^2}, & x > 10; \\ 0, & x \le 10. \end{cases}$$

   (a) Find $P(X > 20)$.

   (b) What is the cumulative distribution function of X?

   (c) What is the probability that, of 6 such types of devices, at least 3 will function for at least 15 hours? What assumptions are you making?

In what follows, we use the round parentheses instead of the curly parentheses.

2. The density function of X is given by

$$f(x) = \begin{cases} a + bx^2, & 0 \le x \le 1; \\ 0, & \text{otherwise}. \end{cases}$$

If $E[X] = 0.75$, find a, b, $E[X^2]$ and $\text{Var}(X)$.

3. Suppose the cumulative distribution function of a random variable X is given by

$$F(x) = \begin{cases} 1 - (x+1)^{-2}, & x > 0; \\ 0, & x \le 0. \end{cases}$$

Evaluate $P(1 < X < 3)$ and $E[X]$.

4. A point is chosen at random on a line segment of length L. Interpret this statement, and find the probability that the ratio of the shorter to the longer segment is less than $1/4$.

5. The annual rainfall (in inches) in a certain region is normally distributed with $\mu = 40$ and $\sigma = 4$.

   (a) What is the probability that, starting with this year, it will take over 10 years before a year occurs having a rainfall of over 50 inches?

   (b) What assumptions are you making?

6. Let X be a normal random variable with mean 12 and variance 4. Find the value of c such that $P(X > c) = .10$.

7. The time (in hours) required to repair a machine is an exponentially distributed random variable with parameter $\lambda = 1/2$. What is

   (a) the probability that a repair time exceeds 2 hours?

   (b) the conditional probability that a repair takes at least 10 hours, given that its duration exceeds 9 hours?

8. Let $Z \sim N(0,1)$. If we define $X = e^{\sigma Z+\mu}$, then we say that X has a log-normal distribution with parameters μ and σ, and we write $X \sim \text{LogNormal}(\mu, \sigma)$.

   (a) If $X \sim \text{LogNormal}(\mu, \sigma)$, find the cumulative distribution function of X in terms of the Φ function.

   (b) Find $E[X]$ and $\text{Var}(X)$.

9. Let X be a random variable that takes on values between 0 and k. That is, $P(0 \leq X \leq k) = 1$. Show that

$$\text{Var}(X) \leq \frac{k^2}{4}$$

   Hint: one approach is to first argue that

$$E[X^2] \leq kE[X]$$

   and then use this inequality to show that

$$\text{Var}(X) \leq k^2 \left(\beta(1-\beta)\right)$$

   where $\beta = \dfrac{E[X]}{k}$.

10. Let $f(x)$ denote the probability density function of a normal random variable with mean μ and variance σ^2. Show that $\mu - \sigma$ and $\mu + \sigma$ are points of inflection of this function. That is, That is, show that the second derivative f'' has an isolated zero and changes sign at $x = \mu - \sigma$ or $x = \mu + \sigma$.

11. Let Z be a standard normal random variable Z, and let g be a differentiable function with derivative g'. To be more precise, assume the growth rate of g at infinities are slower than exponential, e.g., g has a polynomial growth rate.

    (a) Show that $E[g'(Z)] = E[Zg(Z)]$.

    (b) Show that $E[Z^{n+1}] = nE[Z^{n-1}]$.

    (c) Find $E[Z^4]$.

12. If X is an exponential random variable with parameter λ, and $k > 0$, show that $kX$is exponential with parameter λ/k.

13. If $X \sim \text{Geometric}(p)$, show that

$$M_X(t) = \frac{pe^t}{1-(1-p)e^t}, \quad \text{if} \quad t < -\ln(1-p),$$

    Hence, verify that

$$E(X) = \frac{1}{p};$$
$$\text{Var}(X) = \frac{1-p}{p^2}.$$

14. If $X \sim \text{Gamma}(\theta, \alpha)$, show that

$$M_X(t) = \left(\frac{1}{1-\theta t}\right)^{\alpha}, \quad \text{if} \quad t < \frac{1}{\theta}$$

Hence, verify that

$$\begin{aligned} E[X] &= \theta\alpha; \\ \text{Var}(X) &= \theta^2\alpha. \end{aligned}$$