Math2040 Tutorial 10

Spectral theorems

- complex spectral theorem: normal $\Leftrightarrow$ there exists orthonormal eigenbasis of V
- real spectral theorem: self-adjoint $\Leftrightarrow$ there exists orthonormal eigenbasis of V

  (where $T \in \mathcal{L}(V)$, and V is a finite dimensional inner product space)

Lecture 15, Example 1. Recall that (Example 4 in Lecture 14) the operator $T \in \mathcal{L}(\mathbb{C}^2)$ defined by $T(x, y) = (2x - 3y, 3x + 2y)$ is normal, so there exists an orthonormal eigenbasis β by the complex spectral theorem.

1. pick an orthonormal basis, say the standard basis $\gamma = \{(1, 0), (0, 1)\}$
2. matrix representation of T under γ is $[T]_\gamma = \begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix}$
3. characteristic polynomial is $p(\lambda) = (2 - \lambda)^2 + 9$
4. two eigenvalues: $\lambda_1 = 2 + 3i$ and $\lambda_2 = 2 - 3i$
5. respective eigenvectors: $v_1 = (i, 1)$ and $v_2 = (-i, 1)$
6. distinct eigenvalues imply orthogonal eigenvectors
7. normalize $\{v_1, v_2\}$ to get an orthonormal eigenbasis

$$\beta = \left\{ \frac{1}{\sqrt{2}}(i, 1), \frac{1}{\sqrt{2}}(-i, 1) \right\}$$

Lecture 15, Example 2. Let $T \in \mathcal{L}(\mathbb{R}^3)$ be a linear operator whose matrix with respect to the standard basis of $\mathbb{R}^3$ is given by $A = \begin{pmatrix} 14 & -13 & 8 \\ -13 & 14 & 8 \\ 8 & 8 & -7 \end{pmatrix}$

1. $[T^*]_\beta = ([T]_\beta)^*$ for standard basis β, A symmetric implies T self-adjoint
2. real spectram theorem implies existence of eigenbasis
3. characteristic polynomial $p(\lambda) = -\lambda^3 + 21\lambda^2 + 297\lambda - 3645 = -(\lambda + 15)(\lambda - 9)(\lambda - 27)$
4. three eigenvalues: $\lambda_1 = -15$, $\lambda_2 = 9$, and $\lambda_3 = 27$
5. respective eigenvectors: $v_1 = (1, 1, -2)$, $v_2 = (1, 1, 1)$, and $v_3 = (1, -1, 0)$
6. again, distinct eigenvalues implie orthogonal eigenvectors
7. normalize $\{v_1, v_2, v_3\}$ to obtain an orthonormal eigenbasis

$$\beta = \left\{ \frac{1}{\sqrt{6}}(1, 1, -2), \frac{1}{\sqrt{3}}(1, 1, 1), \frac{1}{\sqrt{2}}(1, -1, 0) \right\}$$