Math2040 Tutorial 9

Self-adjoint and normal

- Riesz representation theorem: if $\varphi \in \mathcal{L}(V, \mathbb{F})$, there exists a unique $u \in V$ such that $\varphi(v) = \langle v, u\rangle$, where u is given by $u = \overline{\varphi(e_1)}e_1 + \cdots + \overline{\varphi(e_n)}e_n$ with orthonormal basis $\{e_1, \ldots, e_n\}$; so, there is a one-one correspondence between vectors in V and linear functionals on V
- adjoint $T^* \in \mathcal{L}(W, V)$ satisfies $\langle Tv, w\rangle = \langle v, T^*w\rangle$ for any $v \in V$ and $w \in W$
- Riesz representation theorem $\Rightarrow$ existence and uniqueness of adjoint
- $(aS + bT)^* = \bar{a}S^* + \bar{b}T^*$, $(T^*)^* = T$, $I^* = I$, and $(ST)^* = T^*S^*$
- orthonormal bases β (of V) and γ (of W) $\Rightarrow$ $[T^*]_\gamma^\beta = ([T]_\beta^\gamma)^*$ (adjoint $\leftrightarrow$ conjugate transpose)
- self-adjoint: $T = T^*$ $\Rightarrow$ real eigenvalues, $\langle Tv, v\rangle \in \mathbb{R}$
- normal: $TT^* = T^*T$ $\Rightarrow$ $\|T^*v\| = \|Tv\|$, conjugate eigenvalues ($Tv = \lambda v \Rightarrow T^*v = \bar{\lambda}v$), orthogonal eigenvectors (of distinct eigenvalues)

Lecture 14, Example 2. Fix some vector $u \in V$ and $x \in W$. Define a linear transformation $T : V \to W$ by $Tv = \langle v, u\rangle\, x$ for all $v \in V$. Find the adjoint $T^* : W \to V$.

By definition, we have $\langle v, T^*w\rangle = \langle Tv, w\rangle$ for any $v \in V$ and $w \in W$. But

$$\langle Tv, w\rangle = \langle \langle v, u\rangle\, x, w\rangle = \langle v, u\rangle \langle x, w\rangle = \left\langle v, \overline{\langle x, w\rangle} u\right\rangle = \langle v, \langle w, x\rangle\, u\rangle\,.$$

Since this is true for all $v \in V$, we conclude that $T^*w = \langle w, x\rangle\, u$ for any $w \in W$.

Lecture 14, Example 3. Let $A \in \mathbf{M}_{m\times n}(\mathbb{F})$. The adjoint of $L_A : \mathbb{F}^n \to \mathbb{F}^m$ is given by $L_{A^*} : \mathbb{F}^m \to \mathbb{F}^n$, where $A^* = \overline{A^t}$ is the conjugate transpose of A.

1. by usual inner product in $\mathbb{F}^m$, $\langle L_Av, w\rangle = w^*Av$, where w^* is the conjugate transpose of w
2. note that $(B^*)^* = B$ and $(Bx)^* = x^*B^*$ (as conjugate transpose)
3. for all $v \in \mathbb{F}^n$ we have $w^*Av = (A^*w)^*v = \langle v, L_{A^*}w\rangle$, so $(L_A)^* = L_{A^*}$

Lecture 14, Example 4. Let $T \in \mathcal{L}(\mathbb{R}^2)$ be the linear operator on $\mathbb{R}^2$ (with the standard dot product) defined by $T(x, y) = (2x - 3y, 3x + 2y)$. Then T is normal but not self-adjoint.

1. pick an orthonormal basis, e.g. the standard (orthonormal) basis, $\beta = \{(1, 0), (0, 1)\}$
2. matrix representation $[T]_\beta = \begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix}$
3. for orthonormal basis, $[T^*]_\beta = ([T]_\beta)^*$, not self-adjoint
4. but normal, as $[TT^*]_\beta = [T]_\beta([T]_\beta)^* = \begin{pmatrix} 13 & 0 \\ 0 & 13 \end{pmatrix} = ([T]_\beta)^*[T]_\beta = [T^*T]_\beta$

Lecture 14, Exercise 6. Let $T \in \mathcal{L}(V, W)$. Prove that

(a) T is injective if and only if T^* is surjective,

(b) T is surjective if and only if T^* is injective,

(c) $\dim \operatorname{range} T^* = \dim \operatorname{range} T$,

(d) $\dim \ker T^* = \dim \ker T + \dim W - \dim V$.

1. (Proposition 5, N14) $\operatorname{range} T^* = (\ker T)^\perp$ and $\ker T^* = (\operatorname{range} T)^\perp$
2. (Proposition 3, N12) $U = \left(U^\perp\right)^\perp$ for any finite dimensional subspace U
3. (Proposition 3+6, N6) T is injective iff $\ker T = \{0\}$ and $\operatorname{range} T^* = V$ iff T^* is surjective
4. (Corollary 4, N12) $\dim U^\perp = \dim V - \dim U$ for any subspace U of V (finite dimensional)
5. (Theorem 7, N6) $\dim V = \dim \ker T + \dim \operatorname{range} T$, $\dim W = \dim \ker T^* + \dim \operatorname{range} T^*$

Lecture 14, Exercise 23. Fix $u, x \in V$. Define $T \in \mathcal{L}(V)$ by $Tv = \langle v, u \rangle x$ for all $v \in V$.

(a) Suppose $\mathbb{F} = \mathbb{R}$. Prove that T is self-adjoint if and only if $\{u, x\}$ is linearly dependent.

(b) Prove that T is normal if and only if $\{u, x\}$ is linearly dependent.

1. $\langle v, T^*w \rangle = \langle Tv, w \rangle = \langle \langle v, u \rangle x, w \rangle = \langle v, u \rangle \langle x, w \rangle = \langle v, \langle w, x \rangle u \rangle \Rightarrow T^*w = \langle w, x \rangle u$
2. trivial for $u = 0$ or $x = 0$, assume $u \neq 0$ and $x \neq 0$
3. $(T - T^*)v = \langle v, u \rangle x - \langle v, x \rangle u$
4. T self-adjoint $\Rightarrow$ non-trivial linear combination of x and u
5. $x = \alpha u \Rightarrow (T - T^*)v = \langle v, u \rangle (\alpha u) - \langle v, \alpha u \rangle u = (\alpha - \overline{\alpha}) \langle v, u \rangle$, always zero in a real space
6. $(TT^* - T^*T)v = \langle \langle v, x \rangle u, u \rangle x - \langle \langle v, u \rangle x, x \rangle u = \langle v, x \rangle \langle u, u \rangle x - \langle v, u \rangle \langle x, x \rangle u$
7. T normal $\Rightarrow$ non-trivial linear combination of x and u
8. $x = \alpha u$ for some $\alpha \in \mathbb{F} \Rightarrow$ left hand side vanishes

Lecture 14, Exercise 26. Let $T \in \mathcal{L}(V)$ be a normal operator and $v, w \in V$ be such that $\|v\| = \|w\| = 2$, $Tv = 3v$, and $Tw = 4w$. Show that $\|T(v + w)\| = 10$.

1. v, w are eigenvectors, with distinct eigenvalues
2. v is orthogonal to w, $3 \langle v, w \rangle = \langle Tv, w \rangle = \langle v, T^*w \rangle = 4 \langle v, w \rangle$ as T normal
3. orthogonality $\Rightarrow$ Pythagoras theorem works
4. as $3v \perp 4w$, so $\|T(v + w)\|^2 = \|3v + 4w\|^2 = \|3v\|^2 + \|4w\|^2$