Math2040 Tutorial 8

Orthonormal basis

- orthonormal $\Leftrightarrow$ unit $\|v_i\| = 1$ and orthogonal $\langle v_i, v_j \rangle = 0$ for $i \neq j$
- $\{e_1, \ldots, e_n\}$ orthonormal basis $\Rightarrow v = \langle v, e_1 \rangle e_1 + \cdots + \langle v, e_n \rangle e_n$ for all v
- Gram-Schmidt process $e_1 = \frac{v_1}{\|v_1\|}$, $e_j = \frac{v_j - \sum_{k<j} \langle v_j, e_k \rangle e_k}{\left\| v_j - \sum_{k<j} \langle v_j, e_k \rangle e_k \right\|}$ for $j > 1$
- there exists orthonormal basis β of V (finite dim., complex) such that $[T]_\beta$ upper-triangular

Lecture 11, Example 6. Let $\beta = \{v_1, \ldots, v_n\}$ be an orthonormal basis of a finite dimensional inner product space $(V, \langle \cdot, \cdot \rangle)$. For any $T \in \mathcal{L}(V)$, the matrix $[T]_\beta$ is given by $([T]_\beta)_{ij} = \langle T v_j, v_i \rangle$.

1. suppose $T v_j = A_{1j} v_i + \cdots + A_{nj} v_n$ for $j = 1, \ldots, n$
2. by definition, we have $([T]_\beta)_{ij} = A_{ij}$
3. on the other hand, by orthonormal basis, we have $A_{ij} = \langle T v_j, v_i \rangle$

Orthogonal complements

- orthogonal complement $U^\perp = \{v \in V : \langle v, u \rangle = 0 \text{ for all } u \in U\}$
- U finite dim. subspace of $V \Rightarrow V = U \oplus U^\perp$ and $U = \left(U^\perp\right)^\perp$

Lecture 12, Example 1. Let $V = \{f : [-1, 1] \to \mathbb{R} \text{ continuous}\}$ be the real vector space equipped with the L^2-inner product $\langle f, g \rangle = \int_{-1}^{1} f(x) g(x)\, dx$. Consider the following subspaces of odd and even functions respectively:

$$\begin{aligned} U_{\text{odd}} &:= \{f \in V : f(-x) = -f(x) \text{ for all } x \in [-1, 1]\}, \\ U_{\text{even}} &:= \{f \in V : f(-x) = f(x) \text{ for all } x \in [-1, 1]\} \end{aligned}.$$

then we have $U_{\text{odd}}^\perp = U_{\text{even}}$.

1. ($U_{\text{odd}}^\perp \subset U_{\text{even}}$) let $f \in U_{\text{odd}}^\perp$, want to show $f \in U_{\text{even}}$, i.e. $f(-x) = f(x)$
2. $f \in U_{\text{odd}}^\perp$ means $\langle f, g \rangle = 0$, i.e. $\int_{-1}^{1} f(x) g(x)\, dx = 0$, for any $g \in U_{\text{odd}}$
3. using change of variable, $\int_{-1}^{1} f(-x) g(x)\, dx = 0$ for any $g \in U_{\text{odd}}$
4. combine to get $\int_{-1}^{1} [f(x) - f(-x)]\, g(x)\, dx = 0$ for any $g \in U_{\text{odd}}$
5. but $f(x) - f(-x) \in U_{\text{odd}}$, take $g(x) = f(x) - f(-x)$, we have $\int_{-1}^{1} [f(x) - f(-x)]^2\, dx = 0$
6. as $f(x)$ and $f(-x)$ continuous, $f(x) - f(-x) = 0$, i.e. $f(x) = f(-x)$, so $f \in U_{\text{even}}$
7. ($U_{\text{odd}}^\perp \supset U_{\text{even}}$) for $f \in U_{\text{even}}$, we have $\langle f, g \rangle = 0$ for any $g \in U_{\text{odd}}$ since f is even but g is odd, so $f \in U_{\text{odd}}^\perp$

Lecture 12, Proposition 2. Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space.

(a) $U^{\perp}$ is a subspace of V for any subset $U \subset V$.

1. $0 \in U^{\perp}$ as $\langle 0, u \rangle = 0$ for any $u \in U$
2. for any $w_1, w_2 \in U^{\perp}$, by the linearity of inner product, we have

$$\langle a_1 w_1 + a_2 w_2, u \rangle = a_1 \langle w_1, u \rangle + a_2 \langle w_2, u \rangle = 0$$

for any $u \in U$, so $a_1 w_1 + a_2 w_2 \in U^{\perp}$

$\Rightarrow$ so $U^{\perp}$ is a subspace of V

remark. $U^{\perp}$ is always a subspace, but U is not necessarily a subspace

(b) $\{0\}^{\perp} = V$ and $V^{\perp} = \{0\}$.

1. since $\langle v, 0 \rangle = 0$ for any $v \in V$, then $V \subset \{0\}^{\perp}$; obviously $\{0\}^{\perp} \subset V$; hence $\{0\}^{\perp} = V$
2. suppose $u \in V^{\perp}$, we have $\langle u, v \rangle = 0$ for any $v \in V$; take $v = u$, then $\langle u, u \rangle = 0$, so $u = 0$; hence $V^{\perp} = \{0\}$

(c) For any subset $U \subset V$, we have $U \cap U^{\perp} \subset \{0\}$.

1. if $U \cap U^{\perp}$ is not empty, then for $v \in U \cap U^{\perp}$, we have $\langle v, v \rangle = 0$, so $v = 0$

(d) For any subsets $U \subset W \subset V$, we have $W^{\perp} \subset U^{\perp}$.

1. for any $w^{\perp} \in W^{\perp}$, so $\left\langle w^{\perp}, w \right\rangle = 0$ for any $w \in W$
2. but $u \in U \subset W$, so $\left\langle w^{\perp}, u \right\rangle = 0$ for any $u \in U$, hence $w^{\perp} \in U^{\perp}$

Lecture 12, Proposition 3. If U is a finite dimensional subspace of an inner product space $(V, \langle \cdot, \cdot \rangle)$, then $V = U \oplus U^{\perp}$. Moreover, we have $U = \left(U^{\perp}\right)^{\perp}$

0. It is shown, in the lecture, that $V = U \oplus U^{\perp}$, we now show that $U = \left(U^{\perp}\right)^{\perp}$
1. ($U \subset \left(U^{\perp}\right)^{\perp}$) for any $u \in U$, we have $\left\langle u, u^{\perp} \right\rangle = \overline{\langle u^{\perp}, u \rangle} = \overline{0} = 0$ for any $u^{\perp} \in U^{\perp}$, so $u \in \left(U^{\perp}\right)^{\perp}$
2. ($U \supset \left(U^{\perp}\right)^{\perp}$) for any $v \in \left(U^{\perp}\right)^{\perp}$, we write $v = u + u^{\perp}$ for some $u \in U$ and $u^{\perp} \in U^{\perp}$
3. $u^{\perp} = 0$ because $0 = \left\langle v, u^{\perp} \right\rangle = \left\langle u, u^{\perp} \right\rangle + \left\langle u^{\perp}, u^{\perp} \right\rangle = \left\langle u^{\perp}, u^{\perp} \right\rangle$, so $v = u \in U$