Math2040 Tutorial 7

Diagonalizability

- $[T]_\beta$ upper-triangular $\Leftrightarrow$ $\mathrm{span}\{v_1, \dots, v_j\}$ is a T-invariant subspace for each $j = 1, \dots, n$
- every operator T on V (finite dim., complex) has a basis β such that $[T]_\beta$ upper triangular
- eigenspace $E_\lambda(T) = \ker(T - \lambda I)$
- T diagonalizable $\Leftrightarrow$ $V = E_{\lambda_1}(T) \oplus \cdots \oplus E_{\lambda_m}(T)$, $\lambda_1, \dots, \lambda_m$ all eigenvalues of T (distinct)

Lecture 10, Example 2. The linear operator $T : \mathbb{F}^3 \to \mathbb{F}^3$ defined by $T(x, y, z) = (2x+y, 5y+3z, 8z)$ is diagonalizable since the matrix of T with respect to the standard basis β of $\mathbb{F}^3$ is

$$[T]_\beta = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 5 & 3 \\ 0 & 0 & 8 \end{pmatrix},$$

which is an upper-triangular matrix, hence has 3 distinct eigenvalues 2, 5, and 8. To find an eigenbasis, we compute the respective eigenspaces to be

$$E_2(T) = \mathrm{span}\{(1, 0, 0)\}, \quad E_5(T) = \mathrm{span}\{(1, 3, 0)\}, \quad E_8(T) = \mathrm{span}\{(1, 6, 6)\}.$$

Hence, an eigenbasis is given by $\gamma = \{(1, 0, 0), (1, 3, 0), (1, 6, 6)\}$ and

$$[T]_\gamma = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 8 \end{pmatrix}.$$

1. n distinct eigenvalues implies n linear independence eigenvectors (c.f. Lecture 8, Proposition 5)
2. form eigenbasis, hence diagonalizable (c.f. Lecture 10, Corollary 8)

Lecture 10, Exercise 7. For each $T \in \mathcal{L}(V)$ below, find an eigenbasis of V with respect to T:

(a) $T \in \mathcal{L}(\mathbb{R}^2)$; $T(x, y) = (-2x + 3y, -10x + 9y)$.

(j) $T \in \mathcal{L}(\mathbf{M}_{2\times 2}(\mathbb{R}))$; $T(A) = A^t + 2(\mathrm{tr}\, A)I$.

Solution.

(a)
- let β be the standard basis of $\mathbb{R}^2$ so $[T]_\beta = \begin{pmatrix} -2 & 3 \\ -10 & 9 \end{pmatrix}$
- $T(x, y) = \lambda(x, y)$ means solving $\begin{pmatrix} -2-\lambda & 3 \\ -10 & 9-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ for non-trivial solutions
- get $\lambda = 3$ for any $(x, y) \in \mathrm{span}\{(3, 5)\}$ and $\lambda = 4$ for any $(x, y) \in \mathrm{span}\{(1, 2)\}$
- form eigenbasis $\gamma = \{(3, 5), (1, 2)\}$ and $[T]_\gamma = \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}$

(j)
- let $\beta = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ so $[T]_\beta = \begin{pmatrix} 3 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 3 \end{pmatrix}$

- get eigenvalues $\lambda_1 = -1$, $\lambda_2 = 1$, and $\lambda_3 = 5$ from $\det([T]_\beta - \lambda I) = (\lambda+1)(\lambda-1)^2(\lambda-5) = 0$
- $E_{-1} = \text{span}\left\{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right\}$, $E_1 = \text{span}\left\{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right\}$, $E_5 = \text{span}\left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right\}$
- take $\gamma = \left\{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right\}$ so $[T]_\gamma = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 5 \end{pmatrix}$

Inner product spaces

- inner product $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{F}$ satisfies (i) linearity, (ii) conjugate symmetry, and (iii) positivity
- linearity in 1^{st} slot + (conjugate) symmetry $\Rightarrow$ (conjugate) linearity in 2^{nd} slot
- inner product describes the geometry of an inner product space
  - (lengths) norm $\|v\| = \sqrt{\langle v, v \rangle}$
  - (angles) inner product itself, orthogonality $\langle u, v \rangle = 0$
  - Pythagoras theorem, Cauchy-Schwarz inequality, triangle inequality, parallelogram law

Lecture 11, Example 1.

(b) The complex vector space $\mathbb{C}^n$ with the inner product defined by

$$\langle (z_1, \ldots, z_n), (w_1, \ldots, w_n) \rangle = z_1 \bar{w}_1 + \cdots + z_n \bar{w}_n$$

is an inner product space.

1. (linearity) write $\vec{z} = (z_1, \ldots, z_n)$, $\vec{v} = (v_1, \ldots, v_n)$, and $\vec{w} = (w_1, \ldots, w_n)$

$$\begin{aligned} \langle a\vec{z} + b\vec{v}, \vec{w} \rangle &= \langle (az_1 + bv_1, \ldots, az_n + bv_n), (w_1, \ldots, w_n) \rangle \\ &= (az_1 + bv_1)\bar{w}_1 + \cdots + (az_n + bv_n)\bar{w}_n \\ &= a(z_1\bar{w}_1 + \cdots + z_n\bar{w}_n) + b(v_1\bar{w}_1 + \cdots + v_n\bar{w}_n) \\ &= a\langle \vec{z}, \vec{w} \rangle + b\langle \vec{v}, \vec{w} \rangle \end{aligned}$$

2. (symmetry)

$$\begin{aligned} \overline{\langle (w_1, \ldots, w_n), (z_1, \ldots, z_n) \rangle} &= \overline{w_1\bar{z}_1 + \cdots + w_n\bar{z}_n} \\ &= z_1\bar{w}_1 + \cdots + z_n\bar{w}_n \\ &= \langle (z_1, \ldots, z_n), (w_1, \ldots, w_n) \rangle \end{aligned}$$

3. (positivity)

$$\begin{aligned} 0 &= \langle (z_1, \ldots, z_n), (z_1, \ldots, z_n) \rangle \\ &= z_1\bar{z}_1 + \cdots + z_n\bar{z}_n \\ &= |z_1|^2 + \cdots + |z_n|^2, \end{aligned}$$

which means $z_1 = \cdots = z_n = 0$

Lecture 11, Example 2. For any positive real number $c_1, \ldots, c_n \in \mathbb{R}$, we can define an inner product on $\mathbb{R}^n$ by

$$\langle (x_1, \ldots, x_n), (y_1, \ldots, y_n) \rangle = c_1x_1y_1 + \cdots + c_nx_ny_n.$$

1. linearity and symmetry are similar to the above
2. (positivity)

$$\begin{aligned} 0 &= \langle (x_1, \ldots, x_n), (x_1, \ldots, x_n) \rangle \\ &= c_1x_1^2 + \cdots + c_nx_n^2, \end{aligned}$$

which means $x_1 = \cdots = x_n = 0$ as $c_1, \ldots, c_n$ are positive

3. scaling without reflection