Math2040 Tutorial 6

T-cyclic Subspaces

- T-cyclic subspace generated by v defined by $U := \operatorname{span}\{v, Tv, T^2v, \dots\}$
- smallest T-invariant subspace of V containing v

Lecture 9, Example 1. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear operator defined by $T(x, y, z) = (-y + z, x + z, 3z)$. To find the T-cyclic subspace generated by $v = (1, 0, 0)$.

1. look for the largest linear independent set $\{v, Tv, \dots, T^kv\}$
2. get $\{v, Tv\} = \{(1, 0, 0), (0, 1, 0)\}$ as $T^2v = (-1, 0, 0) = -v$
3. check that $\operatorname{span}\{v, Tv\}$ is T-invariant as $T(av + bTv) = aTv + bT^2v = -bv + aTv$

Hence, we have $U = \operatorname{span}\{v, Tv, T^2v, \dots\} = \operatorname{span}\{v, Tv\} = \{(x, y, 0) : x, y \in \mathbb{R}\}$.

Cayley-Hamilton Theorem

- characteristic polynomial $p(x) := \det([T]_\beta - xI)$
- $p(x)$ does not depend on the choice of β even though $[T]_\beta$ does
- Cayley-Hamilton theorem tells that $p(T) = T_0$

Lecture 9, Exercise 13. Let $T \in \mathcal{L}(V)$ and U be a T-invariant subspace of V. Prove that U is $p(T)$-invariant for any $p(z) \in \mathcal{P}(\mathbb{F})$.

1. by induction, $T^nu \in U$ whenever $u \in U$ for $n \geq 0$
2. let $p(x) = a_0 + a_1x + \cdots + a_mx^m$, so $p(T) = a_0I + a_1T + \cdots + a_mT^m$
3. for any $u \in U$, $p(T)u = a_0u + a_1Tu + \cdots + a_mT^mu$ but $T^nu \in U$ for $n = 0, 1, \dots, m$
4. $p(T)u$ is a linear combination of vectors in U and U is a subspace, so $p(T)u \in U$

Lecture 9, Exercise 17. Let $A \in \mathbf{M}_{n\times n}(\mathbb{F})$. Prove that $\dim \operatorname{span}\{I, A, A^2, \dots\} \leq n$.

1. let β be a basis of V and $n = \dim V$
2. A induces a linear operator T on V by $[T]_\beta^\beta = A$ (or $[T]_\beta = A$)
3. let $p(x)$ be the characteristic polynomial of T
4. Cayley-Hamilton Theorem implies $p(T) = T_0$, say $a_0I + a_1T + \cdots + a_nT^n = T_0$
5. represent using β, we have $a_0I + a_1[T]_\beta + \cdots + a_n\left([T]_\beta\right)^n = O$
6. imply $a_0I + a_1A + \cdots + a_{n-1}A^{n-1} + a_nA^n = O$, or
$$A^n = -\frac{1}{a_n}\left(a_0I + a_1A + \cdots + a_{n-1}A^{n-1}\right) \in \operatorname{span}\{I, A, A^2, \dots, A^{n-1}\}$$
7. by induction $A^m \in \operatorname{span}\{I, A, A^2, \dots, A^{n-1}\}$ for $m \geq n$
8. conclude $\dim \operatorname{span}\{I, A, A^2, \dots\} = \dim \operatorname{span}\{I, A, A^2, \dots, A^{n-1}\} \leq n$

Lecture 9, Exercise 19. Let $T \in \mathcal{L}(V)$ where V is a finite dimensional vector space over $\mathbb{F}$ with $\dim V > 1$. Prove that $\{p(T) : p(z) \in \mathcal{P}(\mathbb{F})\} \neq \mathcal{L}(V)$.

1. let β be a basis of V and $n = \dim V$
2. write $[p(T)]_\beta^\beta = p(A)$ where $A = [T]_\beta$
3. observe that $\{p(T) : p(z) \in \mathcal{P}(\mathbb{F})\}$ is isomorphic to $\{p(A) : p(z) \in \mathcal{P}(\mathbb{F})\}$
4. note that $\{p(A) : p(z) \in \mathcal{P}(\mathbb{F})\} = \operatorname{span}\{I, A, A^2, \dots\}$
5. by previous exercise, $\dim \operatorname{span}\{I, A, A^2, \dots\} \leq n$ and so is $\{p(T) : p(z) \in \mathcal{P}(\mathbb{F})\}$
6. but $\dim \mathcal{L}(V) = n^2 > n$ for $n = \dim V > 1$
7. so $\{p(T) : p(z) \in \mathcal{P}(\mathbb{F})\} \neq \mathcal{L}(V)$ because of different dimensions