Math2040 Tutorial 5

Invertibility and Isomorphisms

- invertible linear maps $T : V \to W$, inverse exists: T^{-1} such that $T^{-1}T = I_V$ and $TT^{-1} = I_W$
- isomorphisms: exists invertible linear maps
- isomorphic if and only if equal dimension (finite, same underlying field)

Lecture 7, Example 7. Find $[T]_\beta$ where $T : \mathbb{R}^2 \to \mathbb{R}^2$ is the reflection about the line $y = 2x$ and β is the standard basis of $\mathbb{R}^2$.

1. describe T using β is non-trivial
2. identify special subspaces, say line $y = 2x$ and line perpendicular to $y = 2x$ ($Tv = \pm v$)
3. pick basis for the subspaces, say $\beta' = \{(1, 2), (-2, 1)\}$
4. by construction get
$$T\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix}, \quad T\begin{pmatrix}-2\\1\end{pmatrix} = -\begin{pmatrix}-2\\1\end{pmatrix}, \quad [T]_{\beta'} = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$$
5. by change of basis
$$[I]^{\beta}_{\beta'} = \begin{pmatrix}1 & -2\\2 & 1\end{pmatrix}, \quad [I]^{\beta'}_{\beta} = \frac{1}{5}\begin{pmatrix}1 & 2\\-2 & 1\end{pmatrix}, \quad [T]_\beta = [I]^{\beta}_{\beta'}[T]_{\beta'}[I]^{\beta'}_{\beta} = \frac{1}{5}\begin{pmatrix}-3 & 4\\4 & 3\end{pmatrix}$$

Lecture 7, Exercise 1. Suppose $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$ are both invertible linear maps. Prove that $ST \in \mathcal{L}(U, W)$ is also invertible and that $(ST)^{-1} = T^{-1}S^{-1}$.

1. exist $T^{-1} : V \to U$ and $S^{-1} : W \to V$ such that $T^{-1}T = I_U$, $TT^{-1} = I_V$, etc.
2. construct $T^{-1}S^{-1}$ and claim to be inverse of ST
3. check $(T^{-1}S^{-1})ST = T^{-1}T = I_U$ and $ST(T^{-1}S^{-1}) = SS^{-1} = I_W$
4. conclude by definition ST is invertible and $(ST)^{-1} = T^{-1}S^{-1}$

Lecture 7, Exercise 20. Let V and W be finite dimensional vector spaces over $\mathbb{F}$ with given bases β and γ respectively. Suppose $T \in \mathcal{L}(V, W)$. Prove that T is invertible if and only if $[T]^\gamma_\beta$ is an invertible matrix. In this case, show that $[T^{-1}]^\beta_\gamma = \left([T]^\gamma_\beta\right)^{-1}$.

1. ($\Rightarrow$) obtain linear map $T^{-1} : W \to V$ by invertibility of T
2. use Proposition 13 to get $[T^{-1}]^\beta_\gamma[T]^\gamma_\beta = [T^{-1}T]^\beta_\beta = I$ and $[T]^\gamma_\beta[T^{-1}]^\beta_\gamma = [TT^{-1}]^\gamma_\gamma = I$
3. conclude $[T]^\gamma_\beta$ invertible and $\left([T]^\gamma_\beta\right)^{-1} = [T^{-1}]^\beta_\gamma$
4. ($\Leftarrow$) suppose we have $Tv = w$, represent by bases $[T]^\gamma_\beta[v]_\beta = [Tv]_\gamma = [w]_\gamma$
5. conclude by invertible matrix $[T]^\gamma_\beta$ that T is bijective and $[T^{-1}]^\beta_\gamma = \left([T]^\gamma_\beta\right)^{-1}$

Invariant Subspaces and Eigenvectors

- goal: decompose a vector space into smaller invariant subspace to understand an operator
- invariant subspace: $T(U) \subset U$, i.e. $Tu \in U$ for all $u \in U$
- eigenvectors and eigenvalues: $Tv = \lambda v$ for some $\lambda \in \mathbb{F}$ and some non-zero $v \in V$

Lecture 8, Example 7. Let $I_V \in \mathcal{L}(V)$ be the identity map on V, i.e. $I_V v = v$ for all $v \in V$. Then every non-zero $v \in V$ is an eigenvector of I_V with eigenvalue 1.

Lecture 8, Example 5. Let $T : \mathbb{F}^2 \to \mathbb{F}^2$ be the linear operator defined by $T(x, y) = (y, 0)$. Then $U = \{(x, 0) : x \in \mathbb{F}\}$ is an invariant subspace under T. However, there does not exist another subspace $W \subset \mathbb{F}^2$ which is invariant under T.

1. check $T(x, 0) = (0, 0) \in U$ and U invariant under T
2. suppose there is some subspace W such that $\mathbb{F}^2 = U \oplus W$
3. argue $\dim W = 1$ and $W = \operatorname{span} w$ for some $w \in \mathbb{F}^2$
4. show w is an eigenvector and $Tw = \lambda w$ as $Tw \in W = \operatorname{span} w$
5. find eigenvectors of T and $(y, 0) = T(x, y) = \lambda(x, y)$ when $(x, y) \in U$ or $(x, y) = (0, 0)$
6. conclude such subspace does not exist

Lecture 8, Exercise 23. Suppose $V = U \oplus W$ where U and W are non-zero subspaces of V. Define the projection onto U to be the operator $P \in \mathcal{L}(V)$ such that $P(u+w) = u$ for any $u \in U$ and $w \in W$. Find all the eigenvalues and eigenvectors of P.

1. check $Pv = \lambda v$ for non-zero v and use direct sum $v = u + w$
2. derive $(1 - \lambda)u = \lambda w$ and $\lambda w = 0 = (1 - \lambda)u$
3. if $w = 0$, then $u \neq 0$ and $\lambda = 1$, so u are eigenvectors of eigenvalues 1
4. if $\lambda = 0$, then $u = 0$, so w are eigenvectors of eigenvalues 0

Lecture 8, Exercise 25. Let $T \in \mathcal{L}(V)$ such that every non-zero vector $v \in V$ is an eigenvector of T. Prove that $T = \lambda I$ for some $\lambda \in \mathbb{F}$.

1. pick any two vectors from V, say v_1 and v_2, with eigenvalues λ_1 and λ_2
2. if $v_2 \in \operatorname{span} v_1$, then $\lambda_2 v_2 = Tv_2 = T(cv_1) = cTv_1 = c\lambda_1 v_1 = \lambda_1 c v_1 = \lambda_1 v_2$
3. otherwise $\lambda(v_1 + v_2) = T(v_1 + v_2) = Tv_1 + Tv_2 = \lambda_1 v_1 + \lambda_2 v_2$
4. by linear independence $(\lambda - \lambda_1)v_1 + (\lambda - \lambda_2)v_2 = 0$ implies $\lambda_1 = \lambda = \lambda_2$
5. conclude every non-zero vector is an eigenvector of the same eigenvalue

***Lecture 8, Exercise 27.** Let $T \in \mathcal{L}(V)$ where V is finite dimensional with $\dim V \geq 3$. Suppose every 2-dimensional subspace $U \subset V$ is invariant under T. Prove that $T = \lambda I$ for some $\lambda \in \mathbb{F}$.

1. show every non-zero vector is an eigenvector and use Exercise 25
2. for every non-zero vector v_0, extend to linearly independent set $\{v_0, v_1, v_2\}$
3. $\operatorname{span}\{v_0, v_1\}$ 2-dimensional subspace, $v_0 \in \operatorname{span}\{v_0, v_1\}$, so $Tv_0 = av_0 + c_1 v_1$
4. $\operatorname{span}\{v_0, v_2\}$ 2-dimensional subspace, $v_0 \in \operatorname{span}\{v_0, v_2\}$, so $Tv_0 = bv_0 + c_2 v_2$
5. use linear independence to get $a = b$ and $c_1 = c_2 = 0$, so $Tv_0 = \lambda v_0$