Math2040 Tutorial 4

Kernels and Ranges

- Let $T \in \mathcal{L}(V, W)$. Then T is injective if and only if $\ker T = \{\mathbf{0}\}$.
- Let $T \in \mathcal{L}(V, W)$. Then T is surjective if and only if $R(T) = W$.
- (Fundamental Theorem of Linear Algebra) If V is finite dimensional, then both $\ker T$ and $R(T)$ are finite dimensional and $\dim V = \dim \ker T + \dim R(T)$.
- If $\dim V = \dim W$, then T is injective if and only if T is surjective.

  (Special case: when T is an operator on V, i.e. $W = V$.)

Lecture 6, Example 10. Let $T : \mathcal{P}_2(\mathbb{R}) \to \mathcal{P}_3(\mathbb{R})$ be the linear map defined by

$$T(f(x)) = 2f'(x) + \int_0^x 3f(t)dt.$$

- Let β be $\{3x, 2 + \frac{3}{2}x^2, 4x + x^3\}$
- Note that β spans $R(T)$ and is linearly independent.
- Then β is a basis of $R(T)$ and $\dim R(T) = \#(\beta) = 3$.
- Since $R(T) \neq W$, T is not surjective.
- Since $\dim \ker T = \dim \mathcal{P}_2(\mathbb{R}) - \dim R(T) = 3 - 3 = 0$, $\ker T = \{\mathbf{0}\}$, i.e. injective.

Lecture 6, Exercise 17. Define a linear map $T : \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$ by $T(f(x)) = \int_0^x f(t)\ dt$. Prove that T is injective but not surjective.

✓ Injective: $T(f(x)) = 0 \Rightarrow \int_0^x f(t)\ dt = 0 \Rightarrow f(x) = 0$.

× Surjective: $1 \in \mathcal{P}(\mathbb{R})$ but $T(f(x)) \neq 1$ for all $f(x) \in \mathcal{P}(\mathbb{R})$.

Lecture 6, Exercise 18. Define a linear map $T : \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$ by $T(f(x)) = f'(x)$. Prove that T is surjective but not injective.

× Injective: $T(x + 1) = 1 = T(x + 2)$ but $x + 1 \neq x + 2$.

✓ Surjective: given any $p(x) \in \mathcal{P}(\mathbb{R})$ let $f(x) = \int_0^x p(t)\ dt$ and $T(f(x)) = p(x)$.

Lecture 6, Exercise 19. Give an example of a linear map $T : \mathbb{F}^\infty \to \mathbb{F}^\infty$ such that (i) T is injective but not surjective; (ii) T is surjective but not injective.

- Forward shift map $(x_1, x_2, x_3, \dots) \mapsto (0, x_1, x_2, \dots)$ is injective but not surjective.
- Backward shift map $(x_1, x_2, x_3, \dots) \mapsto (x_2, x_3, x_4, \dots)$ is surjective but not injective.

Lecture 7, Example 5. Show that for each polynomial $q(x) \in \mathcal{P}(\mathbb{R})$, there exists a polynomial $p(x) \in \mathcal{P}(\mathbb{R})$ such that $[(x^2+5x+7)p(x)]'' = q(x)$.

- Goal: show that the map $p(x) \mapsto [(x^2+5x+7)p(x)]''$ is surjective.
- Linear: multiplication by x^2+5x+7 and differentiations.
- Cannot apply propositions directly as $\mathcal{P}(\mathbb{R})$ is infinite dimensional.
- Fix $q(x) \in \mathcal{P}(\mathbb{R})$ (of degree m).
- Restrict T to $\mathcal{P}_m(\mathbb{R})$ (as an operator on $\mathcal{P}_m(\mathbb{R})$).
- T: linear operator on finite dimensional vector space $\Rightarrow$ surjective if and only if injective.
- Injective: $[(x^2+5x+7)p(x)]'' = 0 \Rightarrow (x^2+5x+7)p(x) = ax+b \Rightarrow p(x) = 0$.

***Lecture 6, Exercise 24.** Supose there exists a linear map $T : V \to V$ such that both $\ker T$ and $R(T)$ are finite dimensional. Prove that V is a finite dimensional.

- Goal: find a finite basis of V.
- Pick a basis of $\ker T$: $\{u_1, \ldots, u_n\}$.
- Pick a basis of $R(T)$: $\{w_1, \ldots, w_m\}$ (and for each w_j there is some v_j such that $Tv_j = w_j$).
- Claim: $\{u_1, \ldots, u_n, v_1, \ldots, v_m\}$ is a basis of V.
- Spanning:
  - for any $v \in V$ we have $Tv \in R(T)$ so $Tv = c_1Tv_1 + \cdots + c_mTv_m$,
  - rearrange to get $T(v - c_1v_1 - \cdots - c_mv_m) = 0$,
  - implies $v - c_1v_1 - \cdots - c_mv_m \in \ker T$ so $v - c_1v_1 - \cdots - c_mv_m = d_1u_1 + \cdots + d_nu_n$,
  - rearrange to get $v = c_1v_1 + \cdots + c_mv_m + d_1u_1 + \cdots + d_nu_n$.
- In fact, it suffices to show $\{u_1, \ldots, u_n, v_1, \ldots, v_m\}$ spans V as it is a finite set.
- Linear independence: (optional, to find the dimension of V)
  - suppose $a_1u_1 + \cdots + a_nu_n + b_1v_1 + \cdots + b_mv_m = 0$,
  - apply linear map T to both sides: $a_1Tu_1 + \cdots + a_nTu_n + b_1Tv_1 + \cdots + b_mTv_m = 0$,
  - get $b_1Tv_1 + \cdots + b_mTv_m = 0$ since $u_i \in \ker T$ (so $Tu_i = 0$),
  - implies $b_1 = \cdots = b_m = 0$ by the linear independence of $\{Tv_1, \ldots, Tv_m\}$,
  - remains $a_1u_1 + \cdots + a_nu_n = 0$,
  - implies $a_1 = \cdots = a_n = 0$ by the linear independence of $\{u_1, \ldots, u_n\}$.