Math2040 Tutorial 3

Basis and Dimension

Lecture 4, Proposition 10. Let V be a finite dimensional vector space over $\mathbb{F}$. Suppose $S = \{v_1, \ldots, v_n\}$ is a subset of V with $\dim V = n$. If either one of the following is satisfied: (1) S is linearly independent, (2) span$S = V$, then S is a basis of V.

Lecture 4, Example 4. Verify that $\{1, (x-5)^2, (x-5)^3\}$ is a basis of the subspace U of $\mathcal{P}_3(\mathbb{R})$ defined by

$$U = \{p \in \mathcal{P}_3(\mathbb{R}) : p'(5) = 0\}.$$

Idea. For any $p \in \mathcal{P}_3(\mathbb{R})$, say $p = a_0 + a_1x + a_2x^2 + a_3x^3$, if $p'(5) = 0$, then we have

$$p'(5) = a_1 + 2a_2(5) + 3a_3(5)^2 = a_1 + 10a_2 + 75a_3 = 0,$$

which implies $a_1 = -10a_2 - 75a_3$. That means

$$U = \{a_0 + (-10a_2 - 75a_3)x + a_2x^2 + a_3x^3 \in \mathcal{P}_3(\mathbb{R}) : a_0, a_2, a_3 \in \mathbb{R}\}.$$

But for any $p \in \text{span}\{1, (x-5)^2, (x-5)^3\}$, say $p = b_1(1) + b_2(x-5)^2 + b_3(x-5)^3$, we have

$$\begin{aligned} p &= (b_1 + 25b_2 - 125b_3) + (-10b_2 + 75b_3)x + (b_2 - 15b_3)x^2 + b_3x^3 \\ &= c_1 + (-10c_2 - 75c_3)x + c_2x^2 + c_3x^3, \end{aligned}$$

where $c_1 = b_1 + 25b_2 - 125b_3$, $c_2 = b_2 - 15b_3$, and $c_3 = b_3$. Essentially, we see that $\{1, (x-5)^2, (x-5)^3\}$ spans U. Also, $\{1, (x-5)^2, (x-5)^3\}$ is obviously linearly independent. Hence, it is a basis.

Linear Maps

Lecture 5, Example 3. The map $T : \mathbb{R}^3 \to \mathbb{R}^2$ is defined by $T(x, y, z) = (2x - 4y + z, x + y + 6z)$ is a linear map.

Idea. (additivity)

$$\begin{aligned} T(x_1 + x_2, y_1 + y_2, z_1 + z_2) &= (2(x_1 + x_2) - 4(y_1 + y_2) + (z_1 + z_2), (x_1 + x_2) + (y_1 + y_2) + 6(z_1 + z_2)) \\ &= (2x_1 - 4y_1 + z_1, x_1 + y_1 + 6z_1) + (2x_2 - 4y_2 + z_2, x_2 + y_2 + 6z_2) \\ &= T(x_1, y_1, z_1) + T(x_2, y_2, z_2) \end{aligned}$$

(homogeneity)

$$\begin{aligned} T(ax, ay, az) &= (2ax - 4ay + az, ax + ay + 6az) \\ &= a(2x - 4y + z, x + y + 6z) \\ &= aT(x, y, z) \end{aligned}$$

Lecture 5, Example 10.

1. Rotation by θ is given by $T_\theta(x, y) = (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)$.

2. Reflection about the x-axis is the linear map $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x, y) = (x, -y)$.

3. Projection on the x-axis is the linear map $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x, y) = (x, 0)$.

Idea. Check directly!

Lecture 5, Exercise 7. Let $T : \mathbb{F}^n \to \mathbb{F}^m$ be a linear map. Show that there exist unique scalars $A_{ij} \in \mathbb{F}$, $i = 1, \ldots, m$ and $j = 1, \ldots, n$ such that $T(x_1, \ldots, x_n) = (A_{11}x_1 + \cdots + A_{1n}x_n, \ldots, A_{m1}x_1 + \cdots + A_{mn}x_n)$.

Idea. Obviously, we let $A_{ij} = (T(e_j))_i$, which is determined uniquely. So, we have $T(e_j) = (A_{1j}, \ldots, A_{mj})$. Therefore, we have the following.

$$\begin{aligned} T(x_1, \ldots, x_n) &= T(x_1e_1 + \cdots + x_ne_n) \\ &= x_1T(e_1) + \cdots + x_nT(e_n) \\ &= (A_{11}x_1 + \cdots + A_{1n}x_n, \ldots, A_{m1}x_1 + \cdots + A_{mn}x_n) \end{aligned}$$

Lecture 5, Exercise 8. Give an example of a function $T : \mathbb{R}^2 \to \mathbb{R}$ such that $T(av) = aTv$ for all $a \in \mathbb{R}$ and all $v \in \mathbb{R}^2$ but $T : \mathbb{R}^2 \to \mathbb{R}$ is not a linear map.

Example. One possible function is $T(x, y) = \begin{cases} x & \text{if } |x| \geq |y| \\ y & \text{otherwise} \end{cases}$, which satisfies $T(av) = aTv$ but fails additivity. For example, we have $T(1, 1) = 1$ and $T(1, 0) + T(0, 1) = 1 + 1 = 2$.

Lecture 5, Exercise 10. Let V and W be vector spaces over the field of rational numbers $\mathbb{Q}$. Suppose $T : V \to W$ is a map such that $T(u + v) = T(u) + T(v)$ for all $u, v \in V$. Prove that T is a linear map.

Idea. By the given assumption, we already have the additivity. So, it remains to show the homogeneity, that is, $T(\alpha v) = \alpha T(v)$ for all $\alpha \in \mathbb{Q}$. Consider $\alpha \in \mathbb{Q}$, we have $\alpha = \frac{p}{q}$ for some $p, q \in \mathbb{Z}$ and $q \neq 0$. Note that $T(-v) = -T(v)$. We want to show that $T(\frac{p}{q}v) = \frac{p}{q}T(v)$, which is equivalent to $qT(\frac{p}{q}v) = pT(v)$.

$$\begin{aligned} qT(\frac{p}{q}v) &= \underbrace{T(\frac{p}{q}v) + \cdots + T(\frac{p}{q}v)}_{\text{sum of } q \text{ terms}} \\ &= T(q \cdot \frac{p}{q}v) \\ &= T(pv) \\ &= T(\underbrace{v + \cdots + v}_{\text{sum of } p \text{ terms}}) \\ &= pT(v) \end{aligned}$$

So, we have established the homogeneity.

Remark. T may not be a linear map over $\mathbb{R}$.