Math2040 Tutorial 2

Linear Independence

Lecture 3, Exercise 1 (a). Is $(1, 2, -3)$ a linear combination of $(-3, 2, 1)$ and $(2, -1, -1)$?

Idea. Suppose
$$(1, 2, -3) = \alpha(-3, 2, 1) + \beta(2, -1, -1).$$
Then we obtain the following system.
$$\begin{cases} -3\alpha + 2\beta = 1 \\ 2\alpha - \beta = 2 \\ \alpha - \beta = -3 \end{cases} \quad \Rightarrow \quad \begin{cases} \alpha = 5 \\ \beta = 8 \end{cases}$$
So, the answer is Yes.

Lecture 3, Exercise 11. Find a number $t \in \mathbb{R}$ such that $\{(3, 1, 4), (2, -3, 5), (5, 9, t)\}$ is linearly dependent in $\mathbb{R}^3$.

Idea. To see if they are linearly dependent, we see if we can find some α, β, and γ (not all zero) such that
$$\alpha(3, 1, 4) + \beta(2, -3, 5) + \gamma(5, 9, t) = (0, 0, 0).$$
In other words, we look for non-trivial solution to the system
$$\begin{pmatrix} 3 & 2 & 5 \\ 1 & -3 & 9 \\ 4 & 5 & t \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$
But there is a non-trvial solution if and only if
$$\det \begin{pmatrix} 3 & 2 & 5 \\ 1 & -3 & 9 \\ 4 & 5 & t \end{pmatrix} = 0 \quad \Leftrightarrow \quad t = 2.$$

Lecture 3, Exercise 15.

1. Consider $\mathbb{C}$ as a real vector space, show that $\{1 + i, 1 - i\}$ is linearly independent.
2. Consider $\mathbb{C}$ as a complex vector space, show that $\{1 + i, 1 - i\}$ is linearly dependent.

Idea. For a real vector space, we look at $\alpha(1 + i) + \beta(1 - i) = 0$ for $\alpha, \beta \in \mathbb{R}$. The only solution would be $\alpha = \beta = 0$. This shows that $\{1 + i, 1 - i\}$ is linearly independent.

For a complex vector space, we do the same thing but for $\alpha, \beta \in \mathbb{C}$. Obviously, we have non-zero α and β (such as $\alpha = 1$ and $\beta = -i$). So, $\{1 + i, 1 - i\}$ is linearly dependent.

From these two examples, we see that the underlying field plays an important role in a vector space.

Lecture 3, Exercise 20. Let $S = \{v_1, \dots, v_m\} \subset V$ be a linearly independent subset of a vector space V over $\mathbb{F}$ and $w \in V$. Prove that if $\{v_1 + w, \dots, v_m + w\}$ is linearly dependent, then $w \in \text{span}\{v_1, \dots, v_m\}$.

Idea.

1. Suppose $S = \{v_1, \dots, v_m\} \subset V$ is linearly independent. This means $\alpha_1 v_1 + \cdots + \alpha_m v_m = 0$ would imply $\alpha_1 = \cdots = \alpha_m = 0$.
2. Assume $\{v_1 + w, \dots, v_m + w\}$ is linearly dependent. This means there is some $\alpha_1, \dots, \alpha_m$ (not all zero) such that $\alpha_1(v_1 + w) + \cdots + \alpha_m(v_m + w) = 0$.
3. Rearrange the equation to get $(\alpha_1 + \cdots + \alpha_m)w = \alpha_1 v_1 + \cdots + \alpha_m v_m$.
4. (a) If $\alpha_1 + \cdots + \alpha_m = 0$, then the equation becomes $\alpha_1 v_1 + \cdots + \alpha_m v_m = 0$. But the first statement implies that $\alpha_1 = \cdots = \alpha_m = 0$, which contradicts with the fact that $\alpha_1, \dots, \alpha_m$ not all zero. So, this is not possible.
   (b) If $\alpha_1 + \cdots + \alpha_m \neq 0$, then we can do division and get $w = \beta_1 v_1 + \cdots + \beta_m v_m$ where $\beta_i = \alpha_i/(\alpha_1 + \cdots + \alpha_m)$.
5. Hence, we have $w \in \text{span}\{v_1, \dots, v_m\}$.

Lecture 3, Example 5. The subset $\{1 + z, z^2, 2 - z + z^2 + 3z^3, z^2 + z^3, 2 - z^2\}$ in $\mathcal{P}_3(\mathbb{F})$ must be linearly dependent since $\{1, z, z^2, z^3\}$ is a subset of 4 vectors in $\mathcal{P}_3(\mathbb{F})$ which span the whole vector space. By Theorem 9 in Lecture 3, any linearly independent subset of $\mathcal{P}_3(\mathbb{F})$ can have at most 4 vectors in it.

Remark. For a finite dimensional vector space, the size of a linearly independent subset must be less than or equal to the size of a subset spanning the vector space (as in Theorem 9 of Lecture 3).

Lecture 3, Exercise 14. Prove or give a counterexample: if $\{v_1, \dots, v_m\}$ and $\{w_1, \dots, w_m\}$ are linearly independent subsets of a vector space V over $\mathbb{F}$, then $\{v_1 + w_1, \dots, v_m + w_m\}$ is also linearly independent.

Counterexample. This is not true in general. One simple example is this: Take $V = \mathbb{R}^2$ and $\mathbb{F} = \mathbb{R}$. Consider $v_1 = (1, 0)$ and $v_2 = (0, 1)$. Similarly, let $w_1 = (0, 1)$ and $w_2 = (1, 0)$. Obviously, $\{v_1 + w_1, \dots, v_m + w_m\} = \{(1, 1), (1, 1)\}$ is linearly dependent.

Lecture 3, Exercise 17. Give an example of a subset $S = \{v_1, v_2, v_3\} \subset \mathbb{R}^3$ that is linearly dependent but none of the three vectors is a scalar multiple of another.

Example. Say v_1 is a linear combination of v_2 and v_3 (since none of the three is a scalar multiple of another). Then we simply choose linearly independent v_2 and v_3 and take v_1 as the sum of the two.

$$\begin{aligned} v_1 &= (1, 1, 0) \\ v_2 &= (1, 0, 0) \\ v_3 &= (0, 1, 0) \end{aligned}$$