Math2040 Tutorial 1

Vector spaces

Lecture 1, Exercise 8. Let $V = \mathbb{R}^2$ and define addition and scalar multiplication as follows:

$$(a_1, a_2) + (b_1, b_2) := (a_1 + b_1, 0)$$
$$\lambda(a_1, a_2) := (\lambda a_1, 0)$$

Is V a real vector space?

Idea. No, V is not a real vector space as it fails to satisfy a property of a vector space, namely the multiplicative identity, where $1v = v$ for all $v \in V$. One could easily find an explicit counterexample that violates the property.

Subspaces

Lecture 2, Exercise 9. Let $\operatorname{tr}(A)$ and $\det(A)$ be the trace and determinant of a square matrix A respectively.

1. Is $\{A \in \mathbf{M}_{n\times n}(\mathbb{F}) : \operatorname{tr}(A) = 0\}$ a subspace of $\mathbf{M}_{n\times n}(\mathbb{F})$?
2. Is $\{A \in \mathbf{M}_{n\times n}(\mathbb{F}) : \operatorname{tr}(A) = 1\}$ a subspace of $\mathbf{M}_{n\times n}(\mathbb{F})$?
3. Is $\{A \in \mathbf{M}_{n\times n}(\mathbb{F}) : \det(A) = 0\}$ a subspace of $\mathbf{M}_{n\times n}(\mathbb{F})$?

Idea. Obviously $\mathbf{M}_{n\times n}(\mathbb{F})$ is a vector space over $\mathbb{F}$. We see if they satisfy the conditions of being a subspace. (For simplicity, we name each subset as U_i.)

1. Yes, it is a subspace (of $\mathbf{M}_{n\times n}(\mathbb{F})$).
   (a) $\mathbf{0}_{n\times n} \in U$ as $\operatorname{tr}(\mathbf{0}_{n\times n}) = 0$.
   (b) If $A, B \in U$, then $\sum_{i=1}^n A_{ii} = 0$ and $\sum_{i=1}^n B_{ii} = 0$. So, $C = A + B \in U$ as
   $$\sum_{i=1}^n C_{ii} = \sum_{i=1}^n (A_{ii} + B_{ii}) = 0.$$
   (c) If $A \in U$ and $\alpha \in \mathbb{F}$, then $\sum_{i=1}^n A_{ii} = 0$. So $D = \alpha A \in U$ as
   $$\sum_{i=1}^n D_{ii} = \sum_{i=1}^n \alpha A_{ii} = \alpha \sum_{i=1}^n A_{ii} = 0.$$
2. No, because $\mathbf{0}_{n\times n} \notin U$ as $\operatorname{tr}(\mathbf{0}_{n\times n}) = 0 \neq 1$.
3. It depends. If $n = 1$, then $U = \{\mathbf{0}_{1\times 1}\}$ is a trivial subspace. If $n \geq 2$, we can find $A, B \in U$ such that $A + B \notin U$.
   $$A = \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

Lecture 2, Exercise 10. Give an example of a subset $U \subset \mathbb{R}^2$ for each of the following:

1. U is closed under addition and under taking additive inverses (i.e. $\mathbf{u} \in U \Rightarrow -\mathbf{u} \in U$) but U is not a subspace.

2. U is closed under scalar multiplication but U is not a subspace.

Idea.

1. $U = \{(a, b) \subset \mathbb{R}^2 : a, b \in \mathbb{Z}\}$ is closed under addition and taking additive inverses as integers are closed under addition and taking additive inverses.

2. $U = \{(x, y) \subset \mathbb{R}^2 : x = 0 \text{ or } y = 0\}$ is closed under scalar multiplication since at least one entry will be zero, and will remain zero under scalar multiplication.

Direct sums

Lecture 2, Exercise 20. Suppose $U = \{(x, x, y, y) \in \mathbb{F}^4 : x, y \in \mathbb{F}\}$. Find a subspace W of $\mathbb{F}^4$ such that $\mathbb{F}^4 = U \oplus W$.

Idea. One possible choice of W is $\{(x, -x, y, -y) \in \mathbb{F}^4 : x, y \in \mathbb{F}\}$. First, it is a subspace.

1. $(0, 0, 0, 0) \in W$.

2. $(x, -x, y, -y) + (w, -w, z, -z) = (x + w, -(x + w), y + z, -(y + z)) \in W$.

3. $\alpha(x, -x, y, -y) = (\alpha x, -(\alpha x), \alpha y, -(\alpha y)) \in W$.

In addition, $\mathbb{F}^4$ is a direct sum of U and W. We need to verify that for each $s \in \mathbb{F}^4$, there is a unique sum $u + v = s$ for $u \in U$ and $v \in W$. Let $u = (x, x, y, y)$ and $v = (w, -w, z, -z)$ and $s = (a, b, c, d)$. So, we need

$$u + v = s \quad \Leftrightarrow \quad \begin{cases} x + w = a \\ x - w = b \\ y + z = c \\ y - z = d \end{cases}$$

and, immediately, we see that x, y, w and z are uniquely determined. Hence, we have $\mathbb{F}^4 = U \oplus W$. (One could also use the propositions in Lecture 2 and verify that $\mathbb{F}^4$ is the same as $U \oplus W$.)

Lecture 2, Example 5. Consider the following subspaces of $\mathbb{F}^3$:

$$U_1 := \{(x, y, 0) \in \mathbb{F}^3 : x, y \in \mathbb{F}\}, \quad U_2 := \{(0, 0, z) \in \mathbb{F}^3 : z \in \mathbb{F}\}, \quad U_3 := \{(0, y, y) \in \mathbb{F}^3 : y \in \mathbb{F}\}.$$

Idea. It is shown in Lecture 2 that the sum $U_1 + U_2 + U_3$ is not a direct sum. A simple example

$$(0, 0, 0) = (0, 1, 0) + (0, 0, 1) + (0, -1, -1)$$

shows that $\mathbf{0}$ cannot be expressed uniquely.

We remark that, as stated in Remark 2 of Lecture 2, the sum $U_1 + U_2 + U_3$ may not be a direct sum despite the fact that the sum of each pair of subspaces is a direct sum. The above example is exactly a counterexample.

$$U_1 \cap U_2 = \{\mathbf{0}\}, \quad U_2 \cap U_3 = \{\mathbf{0}\}, \quad U_3 \cap U_1 = \{\mathbf{0}\}$$