THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2050B Mathematical Analysis I (Fall 2016)
Homework 2 Suggested Solutions to Starred Questions

2. Let
$$S := \{\frac{1}{n} - \frac{1}{m} : m, n \in \mathbb{N}\}.$$
Find $\max S, \sup S, \min S, \inf S$, if they exist; Give your reasoning (including your nonexistence claim).

We recall the definition of maximum (minimum):

Definition 1. *Let S be a nonempty subset of real numbers. We say $M \in \mathbb{R}$ is (the) maximum of S if both of the followings are true:*

(a) $M \in S$.

(b) M is an upper bound for S, namely, for any $s \in S$, we have $s \leq M$.

Similarly, we define minimum:

Definition 2. *Let S be a nonempty subset of real numbers. We say $m \in \mathbb{R}$ is (the) minimum of S if both of the followings are true:*

(a) $m \in S$.

(b) m is a lower bound for S, namely, for any $s \in S$, we have $s \geq m$.

It follows easily from definition that maxima and minima are unique.

Solution:

We claim that

(a) $\sup S = 1$.

(b) $\max S$ does not exist.

(c) $\inf S = -1$.

(d) $\min S$ does not exist.

Proof. (a) We prove that $\sup S = 1$: First we show that 1 is an upper bound for S. Let $m, n \in \mathbb{N}$ be arbitrary. Note that $n \geq 1, m > 0$, hence $\frac{1}{n} \leq 1$, and $\frac{1}{m} > 0$ so that $-\frac{1}{m} < 0$. This gives:

$$\frac{1}{n} - \frac{1}{m} < 1 - 0 = 1. \qquad (*)$$

To show that 1 is the least upper bound, we use the useful criterion: $l \leq \sup S$ if for any $\epsilon > 0$, there exists $s \in S$ such that $s + \epsilon > l$.

Now let $\epsilon > 0$ be given. By Archimedean Property, there is some $m_0 \in \mathbb{N}$ such that $m_0 > \frac{1}{\epsilon}$, whence $\epsilon > \frac{1}{m_0}$. Take $n = 1, m = m_0$, thus $s = \frac{1}{1} - \frac{1}{m_0} = 1 - \frac{1}{m_0} \in S$ so that we have $s + \epsilon = 1 - \frac{1}{m_0} + \epsilon > 1$ by construction. Hence $\sup S = 1$.

(b) $\max S$ does not exist.

We will prove by contradiction. Suppose S had a maximum $M \in \mathbb{R}$. Then M is an upper bound for S. Since 1 is the least upper bound for S, we have $1 \leq M$. However, we see from (*) that for any $s \in S$, $s < 1$. Since the maximum $M \in S$, we have $M < 1$, which is a contradiction. Hence S does not have a maximum.

(c) $\inf S = -1$.

One may use similar arguments as in (a) to prove this, by the symmetry of m, n. We present here a different approach which exploits the symmetry of the set S:

For each $S \subseteq \mathbb{R}$, we denote $-S := \{-x : x \in S\}$. In this question, S is symmetric in the sense that $S = -S$, which may be easily verified. Then our conclusion follows from the proposition below:

Proposition 1. *Let S be a nonempty set of real numbers which is bounded above and below. Then:*

i. $\inf S = -\sup(-S)$

ii. $\sup S = -\inf(-S)$

iii. $\min S = -\max(-S)$, *provided that* $\max(-S)$ *exists in* $\mathbb{R}$.

iv. $\max S = -\min(-S)$, *provided that* $\min(-S)$ *exists in* $\mathbb{R}$.

We prove (i) only. The others are similar.

Proof. Let $a = \inf S$, $b = \sup(-S)$.

- "$a \leq -b$": Let $\epsilon > 0$ be arbitrary. We aim to show that there is $s \in S$ such that $s - \epsilon < -b$. Since $b = \sup(-S)$, for the same ϵ, there is $t \in -S$ such that $t + \epsilon > b$. But $t \in -S$, hence we let $s := -t \in S$, and thus $s - \epsilon = -t - \epsilon < -b$. Since $\epsilon > 0$ is arbitrary, we have $a \leq -b$.
- "$a \geq -b$": Let $u \in S$ be arbitrary. We aim to show that $u \geq -b$. Since $u \in S$, we have $-u \in -S$. Since b is an upper bound for $-S$, we have $b \geq -u$, whence $u \geq -b$. Since $u \in S$ is arbitrary, we have $a \geq -b$.

□

By the proposition, we have: $\inf S = -\sup(-S) = -\sup S = -1$. (Recall that S is 'symmetric')

(d) $\min S$ does not exist.

Suppose it had a minimum $m \in \mathbb{R}$. Then we would have $\max S = \max(-S) = -\min S = -m$. But we have just shown that $\max S$ does not exist. This is a contradiction, and hence $\min S$ does not exist.

□

3. Let f, g be real valued functions on X which are bounded above. Show that

$$\sup\{f(x)+g(x) : x \in X\} \leq \sup\{f(x) : x \in X\} + \sup\{g(x) : x \in X\}$$

or in convenient notations,

$$\sup_{x\in X}(f(x)+g(x)) \leq \sup_{x\in X} f(x) + \sup_{x\in X} g(x).$$

Can strict inequality or equality happen?

Proof. Our first observation is that all the 3 suprema exist in $\mathbb{R}$, because f, g are bounded above.

It suffices to show that for any $y \in \{f(x)+g(x) : x \in X\}$, $y \leq \sup\{f(x) : x \in X\} + \sup\{g(x) : x \in X\}$.

Let $y \in \{f(x)+g(x) : x \in X\}$ be arbitrary. Then there is $x_0 \in X$ such that $y = f(x_0)+g(x_0)$. Observe that by definition, $f(x_0) \leq \sup\{f(x) : x \in X\}$, and that $g(x_0) \leq \sup\{g(x) : x \in X\}$. Hence $y \leq \sup\{f(x) : x \in X\} + \sup\{g(x) : x \in X\}$. Since y is arbitrary, we have:

$$\sup\{f(x)+g(x) : x \in X\} \leq \sup\{f(x) : x \in X\} + \sup\{g(x) : x \in X\}$$

□

Example 1. *(strict inequality)*

$X = [0,1]$, $f(x) = x$, $g(x) = -x$. Then $\sup\{f(x) : x \in X\} = 1$, $\sup\{g(x) : x \in X\} = 0$, and that $\sup\{f(x) : x \in X\} + \sup\{g(x) : x \in X\} = 1$. However, $f(x)+g(x) = 0$, so that $\sup\{f(x)+g(x) : x \in X\} = 0$.

Example 2. *(equality)*

$X = [0,1]$, $f(x) = 1$, $g(x) = -1$. Then $\sup\{f(x) : x \in X\} = 1$, $\sup\{g(x) : x \in X\} = -1$, and that $\sup\{f(x) : x \in X\} + \sup\{g(x) : x \in X\} = 0$. On the other hand, $f(x)+g(x) = 0$, so that $\sup\{f(x)+g(x) : x \in X\} = 0$.

4. Let (x_n) be a real sequence converging to $x \in \mathbb{R}$. Show by ϵ-N definition that
   (a) $\lim_{n\to\infty} |x_n| = |x|$
   (b) If $\alpha < x < \beta$ then there exists $N \in \mathbb{N}$ such that for any $n \geq N$, $\alpha < x_n < \beta$.

Proof. (a) Let $\epsilon > 0$. Since x_n converges to x, there is $N \in \mathbb{N}$ such that for $n \geq N$, $|x_n - x| < \epsilon$. Now with the same N, for $n \geq N$, we have, by triangle inequality, that

$$\big||x_n| - |x|\big| \leq |x_n - x| < \epsilon$$

Hence $|x_n|$ converges to $|x|$.

(b) Since $\alpha < x < \beta$, we let $\epsilon_0 := \min\{\beta - x, x - \alpha\} > 0$. For this ϵ_0, by definition of convergence, there is $N \in \mathbb{N}$ such that for $n \geq N$,

$$|x_n - x| < \epsilon_0$$

On the one hand, $x_n - x < \epsilon_0 \leq \beta - x$. Thus $x_n < \beta$ for $n \geq N$. On the other hand, for $n \geq N$, $x_n - x > -\epsilon_0 \geq -(x - \alpha)$, whence $x_n > \alpha$. Hence $\alpha < x_n < \beta$ for $n \geq N$.

Remark: In the proof for (b) we could also let $\epsilon_0 := \frac{\min\{\beta - x, x - \alpha\}}{2} > 0$, which would make the calculation slightly harder. However, the advantage is that it is always safer to use a smaller epsilon in general, since one may get into trouble obtaining only $\geq$ instead of $>$ in some other cases.

$\square$