MATH1050/1058 Proof-writing Exercise 10

Advice.

- All of the questions are concerned with proofs and dis-proofs on statements about functions in a more abstract and/or theoretical setting, with emphasis on the notions of compositions, surjectivity, injectivity, inverse functions.

  Study the Handouts *Surjectivity and Injectivity*, *Compositions, Surjectivity and Injectivity*, *Notion of inverse functions*, *Relations and the formal definition for the notion of functions*, *Existence and uniqueness of inverse functions*

- Besides the handouts mentioned above, the questions in Assignment 9 and Assignment 10 may also be relevant.

1. (a) Prove each of the statements below:

   i. *Let A, B, C be sets, and $f : A \longrightarrow B$, $g : B \longrightarrow C$ be functions. Suppose $g \circ f$ is surjective. Then g is surjective.*

   ii. *Let A, B, C be sets, and $f : A \longrightarrow B$, $g : B \longrightarrow C$ be functions. Suppose $g \circ f$ is injective. Then f is injective.*

   (b)♣ Let I, J, K be sets, and $\alpha : I \longrightarrow J$, $\beta : J \longrightarrow K$, $\gamma : K \longrightarrow I$ be functions. Suppose $\gamma \circ \beta \circ \alpha$, $\alpha \circ \gamma \circ \beta$ are both injective. Further suppose $\beta \circ \alpha \circ \gamma$ is surjective.

   Prove that each of the functions α, β, γ is both surjective and injective.

2. Dis-prove each of the statements below by giving an appropriate argument.

   (a) *Let A, B, C be sets, and $f : A \longrightarrow B$, $g : B \longrightarrow C$ be functions. Suppose $g \circ f$ is surjective. Then f is surjective.*

   (b) *Let A, B, C be sets, and $f : A \longrightarrow B$, $g : B \longrightarrow C$ be functions. Suppose $g \circ f$ is injective. Then g is injective.*

3. (a) Explain the word *inverse function* by stating an appropriate definition.

   (b) Take for granted the result ($\sharp$) below:

   ($\sharp$) *Suppose A, B are sets, and $f : A \longrightarrow B$ is a function. Then the statements $(\sharp_1), (\sharp_2)$ are logically equivalent:*

   ($\sharp_1$) *f is bijective.*

   ($\sharp_2$) *There exists some unique bijective function $g : B \longrightarrow A$ such that g is an inverse function of f.*

   Remark on convention. We denote the function g described in ($\sharp_2$) by f^{-1}.

   Prove the statements below, with reference to the definitions of surjectivity, injectivity, and inverse functions, and with the help of ($\sharp$) where necessarily:

   i. *Let A, B be sets, and $f : A \longrightarrow B$, $g : B \longrightarrow A$, $h : B \longrightarrow A$ be functions. Supppose $g \circ f = \mathsf{id}_A$ and $f \circ h = \mathsf{id}_B$. Then f is bijective, and $f^{-1} = g = h$.*

   ii. *Let A, B be sets, and $f : A \longrightarrow B$ be a function. Suppose f is bijective. Then f^{-1} is bijective, and $(f^{-1})^{-1} = f$.*

   iii. *Let A, B, C be sets, and $f : A \longrightarrow B$, $g : B \longrightarrow C$ be functions. Suppose f, g are bijective. Then $g \circ f$ is a bijective function, and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.*

4.♢ We introduce this definition for the notion of *union of functions* below:

   Let A, B, C, D be sets, and $f : A \longrightarrow C$, $g : B \longrightarrow D$ be functions. Suppose $f(x) = g(x)$ for any $x \in A \cap B$.

   Define the function $f \cup g : A \cup B \longrightarrow C \cup D$ by $(f \cup g)(x) = \begin{cases} f(x) & \text{if} \quad x \in A \\ g(x) & \text{if} \quad x \in B \end{cases}$.

   The function $f \cup g$ is called the **union of the functions** *f, g.*

   Consider each of the statements below. Determine whether it is true or false. Justify your answer with an appropriate argument.

   (a) *Let A, B, C, D be sets, and $f : A \longrightarrow C$, $g : B \longrightarrow D$ be functions. Suppose $f(x) = g(x)$ for any $x \in A \cap B$. Suppose f, g are surjective. Then $f \cup g : A \cup B \longrightarrow C \cup D$ is surjective.*

   (b) *Let A, B, C, D be sets, and $f : A \longrightarrow C$, $g : B \longrightarrow D$ be functions. Suppose $f(x) = g(x)$ for any $x \in A \cap B$. Suppose f, g are injective. Then $f \cup g : A \cup B \longrightarrow C \cup D$ is injective.*

5.$^\diamondsuit$ Let $J = [0, +\infty)$, and $\sigma : \mathbb{R} \longrightarrow J$ be the function defined by $\sigma(x) = x^2$ for any $x \in \mathbb{R}$.

Prove the statement ($\sharp$):

($\sharp$) *Let A be a set. For any function $g : \mathbb{R} \longrightarrow A$, there exists some unique function $f : J \longrightarrow A$ such that $f \circ \sigma = g$.*

6.$^\clubsuit$ Determine whether the statement ($\sharp$) is true. Justify your answer with an appropriate argument:

($\sharp$) *Let A, B be sets, and $f : A \longrightarrow B, g : A \longrightarrow B$ be functions. Suppose $\Phi : \mathsf{Map}(B, A) \longrightarrow \mathsf{Map}(A, B)$ is the function defined by $\Phi(p) = g \circ p \circ f$ for any $p \in \mathsf{Map}(B, A)$. Further suppose f is surjective and g is injective. Then Φ is injective.*

7. Let $C = \{(x, y) \mid x \in \mathbb{R} \text{ and } y \in \mathbb{R} \text{ and } 9x^2 + 16y^2 = 144\}$.

(a) Let $A = [0, 4]$, $B = [0, 3]$, and $F = C \cap (A \times B)$. Define $f = (A, B, F)$.
Verify that f is a function.

(b) Let $A = [2, 3]$, $B = [-1, 4]$, and $F = C \cap (A \times B)$. Define $f = (A, B, F)$.
Is f a function? Justify your answer.

(c) Let $A = [1, 4]$, $B = [0, 5/2]$, and $F = C \cap (A \times B)$. Define $f = (A, B, F)$.
Is f a function? Justify your answer.

8. Let $I = (0, +\infty)$. (So by definition, $I^2 = \{(x, y) \mid x, y \in \mathbb{R} \text{ and } x > 0 \text{ and } y > 0\}$.)

Let $F = \left\{ (p, z) \,\middle|\, \begin{array}{l} z \in I \text{ and} \\ \text{there exist some } x, y \in I \text{ such that } p = (x, y) \text{ and } z^2 + 2(x + y)z = x^2 + y^2 \end{array} \right\}$.

Define $f = (I^2, I, F)$.

(a)$^\diamondsuit$ Prove that f is a function from I^2 to I with graph F.

(b) i. Write down the explicit 'formula of definition' for the function f.

ii.$^\diamondsuit$ Is f surjective? Justify your answer with reference to the definition of surjectivity.

iii.$^\diamondsuit$ Is f injective? Justify your answer with reference to the definition of injectivity.

9.$^\clubsuit$ In this question, take for granted the validity of the **Intermediate Value Theorem**:

Let $a, b \in \mathbb{R}$, with $a < b$. Let $h : [a, b] \longrightarrow \mathbb{R}$ be a function. Suppose $h(a) \neq h(b)$. Suppose h is continuous on $[a, b]$. Then, for any $\gamma \in \mathbb{R}$, if γ is strictly between $h(a)$ and $h(b)$ then there exists some $c \in \mathbb{R}$ such that $a < c < b$ and $h(c) = \gamma$.

Also take for granted that every polynomial function is continuous on $\mathbb{R}$.

Let $A = (0, 1)$, $B = (0, 1)$, and $F = \{(x, y) \mid x \in A \text{ and } y \in B \text{ and } y^4 + 2y^2 + y = x^5 + x^3 + x^2 + x\}$.

Define $f = (A, B, F)$.

(a) Verify that f is a function from A to B with graph F.

(b) Verify that f is surjective.

(c) Verify that f is injective.

10. We recall/introduce the notions of *constant functions* and *non-constant functions*.

Let A, B be sets, and $g : A \longrightarrow B$ be a function.

- *We say that g is a* **constant function** *if $g(x) = g(w)$ for any $x, w \in A$.*
- *We say that g is a* **non-constant function** *if g is not a constant function.*

Let $f : \mathbb{Q} \longrightarrow \mathbb{R}$ be a function. Suppose that for any $x, y \in \mathbb{Q}$, $f(x + y) = f(x) + f(y)$.

(a) Prove that $f(0) = 0$.

(b)$^\diamondsuit$ Prove that $f(nx) = nf(x)$ for any $x \in \mathbb{Q}$, for any $n \in \mathbb{Z}$.

(c)$^\diamondsuit$ Prove that the statements ($\sharp$), ($\natural$), ($\flat$) are logically equivalent:

($\sharp$) *f is injective.*

($\natural$) *f is non-constant.*

($\flat$) *There exists some $a \in \mathbb{R}\backslash\{0\}$ such that for any $x \in \mathbb{Q}$, $f(x) = ax$.*