MATH1050/1058 Proof-writing Exercise 8

Advice.

- Almost all the questions are concerned with wholesale refutations.
- Study the Handout *Dis-proofs by wholesale refutation* before answering the questions.

  Also, make sure you know what it means (and what it takes) to correctly obtain the negation of a statement (which may involve one or more several quantifiers). Refer to the Handouts *Basics of logic in mathematics*, *Universal quantifier and existential quantifier*, *Statements with several quantifiers* on this matter.
- Besides the handout mentioned above, Question (5) of Assignment 7 is also suggestive on what it takes to give a correct wholesale-refutation argument.
- Sometimes you may want to apply the method of proof-by-contradiction within one passage of a proof.

  In this situation, it may be good to start that passage with the words '*we want to verify blah-blah-blah with the method of proof-by-contradiction* '.

  Be reminded that the assumptions used in such a passage of argument (which you hope will lead to a desired contradiction within that passage) must be stated clearly at the beginning of the passage concerned.

1. Dis-prove the statements below:

   (a) *There exists some* $x \in \mathbb{R}$ *such that* $x^2 + 2x + 3 < 0$.

   (b) *There exist some* $x, y \in \mathbb{R}\backslash\{0\}$ *such that* $(x+y)^2 = x^2 + y^2$.

   (c)$^\diamondsuit$ *There exists some* $r \in \mathbb{R}$ *such that* $r < r^5 \leq r^3$.

   (d)$^\diamondsuit$ *There exist some* $\zeta \in \mathbb{C}\backslash\{1\}$, $n \in \mathbb{N}\backslash\{0,1\}$ *such that* ζ *is an* $(n+1)$*-th root of unity and* ζ *is an* (n^2+n+1)*-th root of unity.*

   (e)$^\diamondsuit$ *There exists some* $s \in \mathbb{Q}$ *such that (for any* $t \in \mathbb{Q}$, $s = 2t+1$*).*

   (f)$^\diamondsuit$ *There exists some* $t \in \mathbb{R}$ *such that (for any* $s \in \mathbb{C}$, $|s| \leq t$*).*

   (g)$^\diamondsuit$ *There exist some* $a \in \mathbb{R}$, $n \in \mathbb{N}\backslash\{0,1,2,3\}$ *such that* $\dfrac{(1+\sqrt{|a|})^n}{n(n-1)(n-2)(n-3)} \leq \dfrac{a^2}{24}$.

2. Dis-prove the statements below. (Various results known as the Triangle Inequality may be useful.)

   (a) *There exists some* $x \in \mathbb{R}$ *such that* $|x+1| > |x| + 1$.

   (b)$^\diamondsuit$ *There exists some* $z \in \mathbb{C}$ *such that* $|z + 3 - 4i| > |z| + 5$.

   (c)$^\diamondsuit$ *There exists some* $x \in \mathbb{R}$ *such that* $|x+4| > 2|x+1| + |x-2|$.

   (d)$^\diamondsuit$ *There exist some* $a, b, c, r, s, t \in \mathbb{R}$ *such that*

   $$\sqrt{(a-r)^2 + (b-s)^2 + (c-t)^2} > \sqrt{(a-1)^2 + (b-2)^2 + (c-3)^2} + \sqrt{(r-1)^2 + (s-2)^2 + (t-3)^2}.$$

   (e)$^\diamondsuit$ *There exists some* $z, w \in \mathbb{C}$ *such that* $w \neq 2z$ *and* $\dfrac{2|z - 2w - 3 - 6i| + 3|w + 2 + 4i|}{|2z - w|} < 1$.

3. Let $a, b, c, d \in \mathbb{C}$. Let $r = 1 + 2(|a| + |b| + |c| + |d|)$.

   (a)$^\clubsuit$ Prove the statement ($\sharp$):

   ($\sharp$) *Let* $\zeta \in \mathbb{C}$. *Suppose* $|\zeta| \geq r$. *Then* $\left|1 + \dfrac{a}{\zeta} + \dfrac{b}{\zeta^2} + \dfrac{c}{\zeta^3} + \dfrac{d}{\zeta^4}\right| \geq \dfrac{1}{2}$.

   (b) Hence, or otherwise, dis-prove the statement ($\natural$):

   ($\natural$) *There exists some* $\alpha \in \mathbb{C}$ *such that* $\alpha^4 + a\alpha^3 + b\alpha^2 + c\alpha + d = 0$ *and* $|\alpha| \geq r$.

4. (a) Dis-prove the statement ($\star$):

   ($\star$) *There exist some positive real numbers* x, y *such that* $(x+y)^2 \leq x^2 + y^2$.

(b)$^\diamondsuit$ Hence, or otherwise, dis-prove the statement $(\star\star)$:

$(\star\star)$ *There exist some positive real numbers u, v such that $\sqrt{u} + \sqrt{v} \leq \sqrt{u + v}$.*

5.$^\clubsuit$ Dis-prove the statement $(\star)$:

$(\star)$ *There exists some $k \in \mathbb{N} \backslash \{0, 1\}$ such that for any positive integer n, the number $k^{1/n}$ is an integer.*

Remark. You will probably need the Well-ordering Principle for Integers.